

HYPOELLIPTICITY FOR INFINITELY DEGENERATE QUASILINEAR EQUATIONS AND THE DIRICHLET PROBLEM

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ABSTRACT. In [9], we considered a class of infinitely degenerate quasilinear equations of the form

$$\operatorname{div} \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) = 0$$

and derived *a priori* bounds for high order derivatives $D^\alpha w$ of their solutions in terms of w and ∇w . We now show that it is possible to obtain bounds just in terms of w for a further subclass of such equations, and we apply the resulting estimates to prove that continuous weak solutions are necessarily smooth. We also obtain existence, uniqueness and interior C^∞ -regularity of solutions for the corresponding Dirichlet problem with continuous boundary data.

1. INTRODUCTION

It is well-known (see [6]) that if A is *elliptic*, and A and b are smooth functions of their arguments, then quasilinear operators in divergence form

$$\mathcal{Q}w = \operatorname{div} A(x, w, \nabla w) + b(x, w, \nabla w)$$

are hypoelliptic: any weak solution w of $\mathcal{Q}w = 0$ is smooth. When \mathcal{Q} is *subelliptic* - i.e., when ellipticity fails only to finite order - then hypoellipticity still holds if \mathcal{Q} is *linear* (see e.g. [13]). When \mathcal{Q} is linear but fails to be subelliptic, the situation is more delicate. For example, Fedî showed in [3] that the two-dimensional operator

$$(1) \quad \partial_x^2 + k(x) \partial_y^2$$

is hypoelliptic if k is smooth and positive for all $x \neq 0$. In this case k is allowed to vanish at any rate at $x = 0$. However, by [7], $\partial_x^2 + k(x) \partial_y^2 + \partial_z^2$ is hypoelliptic in \mathbb{R}^3 if and only if $\lim_{x \rightarrow 0} x \log k(x) = 0$. A quasilinear version of operators of the form (1) arises when one considers two-dimensional Monge-Ampère equations

$$(2) \quad u_{ss} u_{tt} - u_{st}^2 = k(s, t), \quad (s, t) \in \tilde{\Omega} \subset \mathbb{R}^2,$$

together with the classical partial Legendre transformation $(x, y) = T(s, t)$ given by

$$(3) \quad \begin{cases} x &= s \\ y &= u_t \end{cases}.$$

Indeed, assuming that T is invertible, (2) and (3) lead to the two-dimensional quasilinear equation

$$(4) \quad \partial_x^2 w + \partial_y \{k(x, w(x, y)) \partial_y w\} = 0, \quad (x, y) \in \Omega = T(\tilde{\Omega}),$$

satisfied in the weak sense by $w(x, y) = t$. In [10] and [11], two of the authors extended Fedî's two-dimensional regularity result for linear equations to certain solutions w of (4) obtained through the transformation (3) from a solution of the Monge-Ampère equation (2). The coefficient k considered there is assumed to satisfy

$$(5) \quad |k_t(s, t)| \leq C k(s, t)^{\frac{3}{2}}, \quad (s, t) \in \tilde{\Omega}.$$

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Thus k is required to become more independent of its second variable as k degenerates. Notice that the coefficient k in (1) is independent of the second variable, and then (5) is automatically true. The main result in [11] establishes that degenerate two-dimensional Monge-Ampère equations (2) with smooth right-hand side k satisfying (5) are hypoelliptic. This was the first known hypoellipticity result for infinitely degenerate Monge-Ampère equations. More general equations than (4) are also treated in the papers above, including the equation for prescribed Gaussian curvature.

2. DESCRIPTION OF THE RESULTS

In the present work, we improve the two-dimensional results above by lowering the exponent $\frac{3}{2}$ in (5) to the optimal exponent 1; this optimality is shown in Section 3 below. We also extend the theory of regularity for degenerate quasilinear equations of the type treated in [10] and [11] to any dimension $n \geq 2$ and to more general quasilinear problems. In this process we have to deal with several fundamental difficulties associated with higher dimensions and the more general structure of the equations. We consider quasilinear equations of the divergence form

$$(6) \quad \mathcal{Q}w = \operatorname{div} \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) = 0 \quad \text{in } \Omega,$$

where Ω is an open bounded connected subset of \mathbb{R}^n and the matrix \mathcal{A} , vector function $\vec{\gamma}$ and scalar function f are smooth functions of their arguments. Here we adopt the vector notation $\vec{u} = (u^1, u^2, \dots, u^n)$; ∇w denotes the gradient $\nabla w = (\partial_1 w, \partial_2 w, \dots, \partial_n w)'$ where $\partial_i = \frac{\partial}{\partial x_i}$ is the i^{th} partial derivative; and the divergence operator applied to a vector function \vec{u} is given by $\operatorname{div} \vec{u} = \partial_1 u^1 + \partial_2 u^2 + \dots + \partial_n u^n$. In our applications, (6) will sometimes be satisfied in the strong sense, i.e. in the pointwise sense for \mathcal{C}^2 functions w , and other times in the weak sense; see Section 7.1.2 in the Appendix for a precise definition. Note that in the case $n = 2$, equation (4) is included among equations of the form $\mathcal{Q}w(x_1, x_2) = 0$ by choosing $\vec{\gamma} = \vec{0}$, $f = 0$ and $\mathcal{A}(x_1, x_2, z)$ to be the diagonal matrix $\operatorname{diag}(1, k(x_1, x_2, z))$ with k independent of x_2 . However, equation (6) does not include systems obtained from the Monge-Ampère equation by the partial Legendre transform introduced in [8] for higher dimensions, and the treatment of such systems when ellipticity fails to infinite order remains a challenging open problem in dimensions bigger than two.

In Section 5, under structural restrictions on \mathcal{A} and $\vec{\gamma}$ which are similar to (5) (although weaker, see Conditions 2.3 and 2.10), we first obtain local a priori bounds for the Lipschitz norm of smooth solutions in terms of their L^∞ norm and the parameters inherent to the equation. This result together with the main theorem in [9] (see Theorem 2.9 below) provides a priori control of *all* derivatives of a smooth solution in terms of the supremum norm of the solution.

In Section 6, we apply the a priori estimates together with an approximation scheme to prove that continuous weak solutions are smooth, and to establish existence, uniqueness, and regularity of solutions of the Dirichlet problem. To do so, we use a class of custom-built barrier functions, a maximum principle and a comparison principle adapted to our class of equations (see Sections 7.4, 7.2 and 7.3 in the Appendix).

The method used to construct the barriers in Lemma 7.6 takes into account only the modulus of continuity of solutions on the boundary. This generalizes most known barrier constructions, which usually require higher regularity of solutions on the boundary.

As already mentioned, one of our main results, Theorem 2.18 below, states that under certain hypotheses on the coefficients \mathcal{A} , every *continuous* weak solution w of (6) is infinitely differentiable, and all of its derivatives are locally controlled by $\|w\|_{L^\infty}$. The conditions imposed on the coefficients allow them to vanish to infinite order, so the quasilinear operator \mathcal{Q} in (6) is not in general uniformly elliptic or even subelliptic. This is the first known hypoellipticity result for infinitely degenerate quasilinear equations in n dimensions.

We now state special cases of the main Theorems 2.17 and 2.18. We include these simpler versions to illustrate the principal features of our results without the technical assumptions of the general case. A *domain* will always mean an open connected set.

Theorem 2.1 (Dirichlet problem). *Let Ω be a strongly convex domain in \mathbb{R}^n containing the origin. Let $k^i(x, z)$, $i = 2, \dots, n$, be smooth nonnegative functions in $\Omega \times \mathbb{R}$ such that*

$$k^i(x, z) > 0 \quad \text{if } x_j \neq 0 \text{ for some } j \neq i$$

(this means that $k^i(x, z)$ may vanish only for those (x, z) so that x lies on the i^{th} -coordinate axis), and such that for some $B > 0$,

$$\left| \frac{\partial}{\partial z} k^i(x, z) \right| \leq B k^*(x, z) \quad \text{for all } (x, z) \in \Omega \times \mathbb{R},$$

where $k^ = \min_{i=2, \dots, n} k^i$. Then, for any continuous function φ on $\partial\Omega$, there exists a unique continuous strong solution w to the Dirichlet problem*

$$\begin{cases} \frac{\partial^2}{\partial x_1^2} w(x) + \sum_{i=2}^n \frac{\partial}{\partial x_i} k^i(x, w(x)) \frac{\partial}{\partial x_i} w(x) = 0 & \text{in } \Omega, \\ w = \varphi & \text{on } \partial\Omega, \end{cases}$$

i.e., there exists a unique w that is both a strong solution of the differential equation in Ω and continuous in $\bar{\Omega}$ with boundary values φ . Moreover, this solution $w \in C^0(\bar{\Omega}) \cap C^\infty(\Omega)$.

Theorem 2.2 (Regularity of solutions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin, and suppose that $k^i(x, z)$, $i = 2, \dots, n$, are as in Theorem 2.1. Then any continuous weak solution w of*

$$\frac{\partial^2}{\partial x_1^2} w(x) + \sum_{i=2}^n \frac{\partial}{\partial x_i} k^i(x, w(x)) \frac{\partial}{\partial x_i} w(x) = 0 \quad \text{in } \Omega$$

is also a strong solution, and satisfies $w \in C^\infty(\Omega)$.

2.1. A priori estimates. For $\tilde{x} \in \mathbb{R}^n$ and $\vec{r} \in \mathbb{R}_+^n$, we denote by $\mathcal{R}(\tilde{x}, \vec{r})$ the box centered at \tilde{x} with edges parallel to the coordinate axes and half-edgelengths given by \vec{r} , i.e.,

$$(7) \quad \mathcal{R}(\tilde{x}, \vec{r}) = [\tilde{x}_1 - r^1, \tilde{x}_1 + r^1] \times \cdots \times [\tilde{x}_n - r^n, \tilde{x}_n + r^n].$$

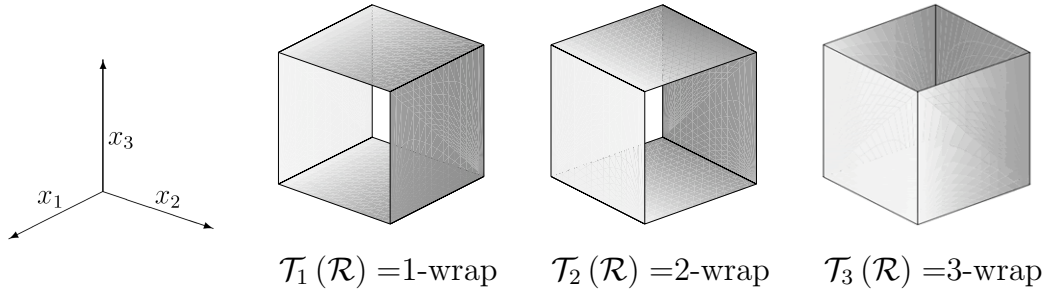
When \tilde{x} is the origin we will just write $\mathcal{R}(\vec{r})$ for $\mathcal{R}(0, \vec{r})$ and we also adopt the summation notation $\mathcal{R}(\tilde{x}, \vec{r}) = \tilde{x} + \mathcal{R}(\vec{r})$. When \vec{r} and \tilde{x} are fixed or clear from context, we will omit them and simply write \mathcal{R} for $\mathcal{R}(\tilde{x}, \vec{r})$. For any positive constant γ , $\gamma\mathcal{R}(\tilde{x}, \vec{r})$ will denote the box centered at \tilde{x} with half-edgelengths given by $\gamma\vec{r}$, i.e.,

$$(8) \quad \gamma\mathcal{R}(\tilde{x}, \vec{r}) = \mathcal{R}(\tilde{x}, \gamma\vec{r}) = \tilde{x} + \mathcal{R}(\gamma\vec{r}).$$

We define the i -wrap $\mathcal{T}_i(\mathcal{R})$ of the box \mathcal{R} as the set of faces of $\partial\mathcal{R}$ containing the direction $\vec{e}_i = (\delta_{ij})_{j=1, \dots, n}$:

$$\mathcal{T}_i(\mathcal{R}) = \overline{\partial\mathcal{R} \setminus \{y : |y_i - \tilde{x}_i| = r^i\}}.$$

The following figure illustrates i -wraps in \mathbb{R}^3 :



We will use the notation $\Gamma = \Omega \times \mathbb{R}$ unless specified to the contrary.

Condition 2.3 (Nondegeneracy condition). *We say that $\vec{k} = (k^1, \dots, k^n)$ satisfies Condition 2.3 in Γ if \vec{k} has continuous second order derivatives in Γ , $\vec{k} \geq 0$ in Γ , and \vec{k} satisfies the following:*

(i) For every subdomain Ω' with $\overline{\Omega'} \subset \Omega$, there exists $c > 0$ such that if $\Gamma' = \Omega' \times \mathbb{R}$, then

$$\inf_{(x,z) \in \Gamma'} \max_{1 \leq i \leq n} k^i(x, z) \geq c > 0.$$

(ii) For all $x \in \Omega$ and $0 < \varepsilon < n^{-\frac{1}{2}} \text{dist}(x, \partial\Omega)$, there exists $\vec{r} = \vec{r}(x, \varepsilon, \vec{k}) = (r^1, \dots, r^n)$, $0 < r^1, \dots, r^n < \varepsilon$, and a box $\mathcal{R} = \mathcal{R}(\tilde{x}, \vec{r})$ so that $x \in \frac{1}{3}\mathcal{R}$ and

$$k^i(y, z) > 0 \text{ whenever } y \in \mathcal{T}_i(\mathcal{R}), \quad z \in \mathbb{R}, \quad i = 1, \dots, n.$$

The restriction on the size of ε ensures that $\mathcal{R} \subset \Omega$.

Note that if $k^1 \equiv 1$, then property (i) in Condition 2.3 is trivially satisfied. If each k^i is nonnegative and has only *isolated zeros*, then property (ii) in Condition 2.3 holds.

Remark 2.4. Property (ii) in Condition 2.3 holds for \vec{k} in $\Gamma = \Omega \times \mathbb{R}$ if and only if for every $x \in \Omega$ and $\varepsilon > 0$, there exists a box \mathcal{R} in Ω with $x \in \frac{1}{3}\mathcal{R}$ and $0 < r^1, \dots, r^n < \varepsilon$ such that for all $y \in \partial\mathcal{R}$ and $z \in \mathbb{R}$, the set of vectors

$$\mathcal{S}(y, z) = \{k^1(y, z)\vec{e}_1, k^2(y, z)\vec{e}_2, \dots, k^n(y, z)\vec{e}_n\}, \quad \vec{e}_i = (\delta_{ij})_{j=1, \dots, n},$$

spans the tangent space to $\partial\mathcal{R}$ at y .

A structural condition that we impose on the matrix \mathcal{A} in (6) is that it is equivalent to a diagonal matrix in the following sense:

Condition 2.5 (Diagonal condition). For \vec{k} as in Condition 2.3, we assume that for some $\Lambda \geq 1$, the matrix \mathcal{A} satisfies

$$(9) \quad \sum_{i=1}^n k^i(x, z) \xi_i^2 \leq \xi^t \mathcal{A}(x, z) \xi \leq \Lambda \sum_{i=1}^n k^i(x, z) \xi_i^2$$

for all $\xi \in \mathbb{R}^n$ and $(x, z) \in \Gamma = \Omega \times \mathbb{R}$.

Remark 2.6. Because of Condition 2.5 or Remark 2.4, we can state property (ii) in the non-degeneracy Condition 2.3 in terms of the matrix \mathcal{A} as follows: For every $x \in \Omega$ and ε with $0 < \varepsilon < n^{-\frac{1}{2}} \text{dist}(x, \partial\Omega)$, there exist $\vec{r} = (r^1, \dots, r^n)$ with $0 < r^1, \dots, r^n < \varepsilon$ and a box $\mathcal{R} = \mathcal{R}(\tilde{x}, \vec{r})$ such that $x \in \frac{1}{3}\mathcal{R}$ and for every $y \in \partial\mathcal{R}$ and nonzero $\vec{v}(y)$ tangent to $\partial\mathcal{R}$ at y ,

$$(10) \quad \vec{v}(y) \cdot \mathcal{A}(y, z) \vec{v}(y) > 0 \quad \text{for all } z \in \mathbb{R}.$$

Recall that a domain Ω is an open connected set. Ω' will always denote a domain with compact closure in Ω ; this will be abbreviated $\Omega' \Subset \Omega$.

Definition 2.7 (Subunit type). We say that a vector field $G = \sum_{i=1}^n \gamma^i(x, z) \frac{\partial}{\partial x_i}$ with bounded coefficients γ^i is of subunit type with respect to \mathcal{A} in $\Gamma = \Omega \times \mathbb{R}$ if for every $\Omega' \Subset \Omega$ and $M_0 \geq 1$, there is a constant $B_\gamma = B_\gamma(\Omega', M_0) > 0$ such that

$$\left(\sum_{i=1}^n \gamma^i(x, z) \xi_i \right)^2 \leq B_\gamma^2 \xi^t \mathcal{A}(x, z) \xi \quad \text{for all } (x, z) \in \Gamma'_{M_0} = \Omega' \times [-M_0, M_0], \xi \in \mathbb{R}^n.$$

We will impose further conditions on \mathcal{A} . To motivate them, we recall the classical inequality of Wirtinger type on a domain $\Phi \subset \mathbb{R}^{n+1}$: if k is nonnegative with bounded second derivatives on Φ , then

$$(11) \quad |\nabla_{x,z} k(x, z)| \leq C \left\{ \|\nabla_{x,z}^2 k\|_{L^\infty(\Phi)}^{\frac{1}{2}} + (\text{dist}((x, z), \partial\Phi))^{-\frac{1}{2}} \right\} \sqrt{k(x, z)}$$

for all $(x, z) \in \Phi$ (see e.g. the appendix in [10]).

Inequality (11) is crucial in our calculations, and although it has an analogue for nonnegative diagonal matrices, it does not extend to general matrix functions.

Definition 2.8 (Subordinate matrix). *We say that \mathcal{A} is subordinate in $\Gamma = \Omega \times \mathbb{R}$ if for every $\Omega' \Subset \Omega$ and $M_0 \geq 1$, there exists $B_{\mathcal{A}} = B_{\mathcal{A}}(\Omega', M_0) > 0$ such that*

$$(12) \quad \sum_{i=1}^n |\partial_i \mathcal{A}(x, z) \xi|^2 + |\partial_z \mathcal{A}(x, z) \xi|^2 \leq B_{\mathcal{A}}^2 \xi^t \mathcal{A}(x, z) \xi$$

for all $\xi \in \mathbb{R}^n$, $(x, z) \in \Gamma'_{M_0}$, where $\Gamma'_{M_0} = \Omega' \times [-M_0, M_0]$.

We always consider locally bounded solutions w , i.e. $\|w\|_{L^\infty(\Omega')} < \infty$ for all subdomains $\Omega' \Subset \Omega$. Thus, we deal only with a solution w whose graph on Ω' is contained in a *bounded* connected set

$$(13) \quad \Gamma'_{M_0} = \Omega' \times [-M_0, M_0] \Subset \Gamma = \Omega \times \mathbb{R}$$

for some $M_0 = M_0(w, \Omega') < \infty$. For convenience we also assume $M_0 \geq 1$.

To obtain our main results, we will use the following a priori estimates obtained in [9] for the class of equations (6).

Theorem 2.9 (Theorem 1.8 in [9]). *Let Ω be a bounded domain in \mathbb{R}^n and $\Gamma = \Omega \times \mathbb{R}$. Let $\vec{k}(x, z) \in \mathcal{C}^2(\Gamma)$ and $\mathcal{A}(x, z)$, $f(x, z)$ and $\vec{\gamma}(x, z) \in \mathcal{C}^\infty(\Gamma)$. Suppose that*

- (i) \mathcal{A} satisfies (9), where $\vec{k}(x, z)$ satisfies the nondegeneracy Condition 2.3 in Γ ,
- (ii) \mathcal{A} is subordinate in Γ (Definition 2.8),
- (iii) $\vec{\gamma}$ is of subunit type with respect to \mathcal{A} in Γ (Definition 2.7).

Then for every smooth solution w of (6) in Ω , integer $N \geq 0$ and subdomains $\Omega'' \Subset \Omega' \Subset \Omega$, there exists a constant $\mathcal{C}_{\|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, N} =$

$$\mathcal{C}_{\|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, N} \left(n, B, \vec{k}, \Lambda, \|\mathcal{A}\|_{\mathcal{C}^{N+2}(\bar{\Gamma})}, \|f\|_{\mathcal{C}^{N+1}(\bar{\Gamma})}, \|\vec{\gamma}\|_{\mathcal{C}^{N+1}(\bar{\Gamma})}, \Omega, \Omega', \Omega'' \right)$$

such that

$$(14) \quad \sum_{|\vec{\alpha}| \leq N} \|D^{\vec{\alpha}} w\|_{L^\infty(\Omega'')} \leq \mathcal{C}_{\|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, N}.$$

Here $\tilde{\Gamma} = \Omega' \times [-2\|w\|_{L^\infty(\Omega')}, 2\|w\|_{L^\infty(\Omega')}]$, and B denotes B_γ , $B_{\mathcal{A}}$.

Our main application of these a priori estimates is the hypoellipticity result stated in Theorem 2.18 for (infinitely degenerate) quasilinear equations of the form (6). In it, as in the special two-dimensional case contained in [10], we will assume extra conditions on the coefficients, namely, that the *nonlinear* and the *infinitely degenerate* characters do not occur simultaneously in the sense described below. We denote

$$(15) \quad k^*(x, z) = \min_{i=1, \dots, n} \{k^i(x, z)\}.$$

Condition 2.10 (Super Subordination condition). *We say that \mathcal{A} satisfies the super subordination condition in $\Gamma = \Omega \times \mathbb{R}$ if for every $\Omega' \Subset \Omega$ and $M_0 \geq 1$, there exist constants $B_{\mathcal{A}} = B_{\mathcal{A}}(\Omega', M_0)$ and $B'_{\mathcal{A}} = B'_{\mathcal{A}}(\Omega', M_0)$ such that if $(x, z) \in \Gamma'_{M_0}$ and $\xi \in \mathbb{R}^n$, then*

$$(16) \quad |\partial_z \mathcal{A}(x, z) \xi|^2 \leq B_{\mathcal{A}}^2 k^*(x, z) \xi^t \mathcal{A}(x, z) \xi,$$

$$(17) \quad \sum_{i=1}^n |\partial_i \partial_z \mathcal{A}(x, z) \xi|^2 + |\partial_z^2 \mathcal{A}(x, z) \xi|^2 \leq (B'_{\mathcal{A}})^2 \xi^t \mathcal{A}(x, z) \xi.$$

If \mathcal{A} is diagonal, then condition (17) follows from (16) by Wirtinger's inequality: see Remark 2.12.

We say $\vec{\gamma}(x, z)$ satisfies the super subordination condition if for all $(x, z) \in \Gamma'_{M_0}$ and $\xi \in \mathbb{R}^n$,

$$(18) \quad |\partial_z \vec{\gamma}(x, z) \cdot \xi|^2 \leq B_\gamma^2 k^*(x, z) \xi^t \mathcal{A}(x, z) \xi,$$

for some $B_\gamma = B_\gamma(\Omega', M_0)$.

The extra vanishing condition (16) on $\partial_z \mathcal{A}$ is a stronger form of the part of (12) involving $\partial_z \mathcal{A}$. In the two-dimensional diagonal case $\mathcal{A} = \text{diag}(1, k)$, inequality (12) always holds for any \mathcal{C}^2 nonnegative $k(x, z)$, and it takes the form (11), while the more restrictive (16) with $k^* = k$ takes the form

$$(19) \quad |\partial_z k(x, z)| \leq B k(x, z)$$

(compare (5)), which does not hold in general for nonnegative $k(x, z)$. On the other hand, if $f(x)$ is any smooth nonnegative function in \mathbb{R}^n and $h(x, z)$ is a nonnegative Lipschitz function in \mathbb{R}^{n+1} , then

$$k(x, z) = f(x) [1 + h(x, z)]$$

satisfies (19). Indeed, we have the following lemma; see Section 6.4 in the appendix of [10] for details.

Lemma 2.11 ([10]). *Let $k(x, z)$ be a smooth nonnegative function in a bounded region $T \subset \mathbb{R}^n \times \mathbb{R}$, and assume that for some $\gamma, B \geq 1$,*

$$(20) \quad |\partial_z k(x, z)| \leq B k(x, z)^\gamma.$$

Then, for every $(x_0, z_0) \in T$, there exists a smooth function $f(x) \geq 0$ and a Lipschitz function $h(x, z)$, with Lipschitz constant depending only on $B, \|k\|_{L^\infty(T)}$ and T , such that

$$(21) \quad k(x, z) = f(x) \left(1 + f(x)^{\gamma-1} h(x, z)\right)$$

for all (x, z) in a neighborhood of (x_0, z_0) . Moreover, $h(x_0, z_0) = 0$. In particular,

$$C^{-1} k(x, z') \leq k(x, z) \leq C k(x, z'), \quad (x, z), (x, z') \in T,$$

where $C = C(B, \text{diam } T)$. Conversely, if $h(x, z)$ is smooth and $f(x)$ is a nonnegative smooth function such that $f(x)^\gamma$ is smooth, then $k(x, z)$ given by (21) is smooth and satisfies (20) for some $B = B(h, f, \text{diam } T)$.

Remark 2.12. *As noted earlier, if \mathcal{A} is diagonal then the second extra vanishing condition (17) follows from the first one (16) by (11). Indeed, it is enough to prove this for a scalar function $k(x, z)$. If (19) holds, then*

$$\tilde{k}(x, z) := \partial_z k(x, z) + B k(x, z) \geq 0,$$

so by (11), for all $(x, z) \in \Gamma'_{M_0}$,

$$\left| \nabla_{x,z} \tilde{k}(x, z) \right| \leq C \left\{ \left\| \nabla_{x,z}^2 \tilde{k} \right\|_{L^\infty(\Gamma'_{2M_0})}^{\frac{1}{2}} + (\text{dist}((x, z), \partial \Gamma'_{2M_0}))^{-\frac{1}{2}} \right\} \sqrt{\tilde{k}(x, z)},$$

where $\Gamma'_{2M_0} = \Omega'' \times [-2M_0, 2M_0]$ with $\Omega' \in \Omega'' \in \Omega$. Hence from (19) we obtain

$$\sum_{i=1}^n |\partial_i \partial_z k|^2 + |\partial_z^2 k|^2 \leq \tilde{B}^2 k(x, z)$$

for a suitable constant \tilde{B} .

Under the extra assumptions in the super subordination Condition 2.10, we will obtain interior a priori control of all derivatives (including first order ones) of smooth solutions in terms of the supremum norm of the solutions:

Theorem 2.13 (A priori estimate). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $\Gamma, \mathcal{A}(x, z)$, $f(x, z)$, $\vec{\gamma}(x, z)$ and $\vec{k}(x, z)$ are as in Theorem 2.9, and also that \mathcal{A} and $\vec{\gamma}$ satisfy the super subordination Condition 2.10 in Γ . If w is a smooth solution of (6) in Ω , then for any nonnegative integer N and subdomain $\Omega' \Subset \Omega$, there exists a constant $\mathcal{C}_{\|w\|_{L^\infty(\Omega')}, N} =$*

$$\mathcal{C}_{\|w\|_{L^\infty(\Omega')}, N} \left(n, B, \vec{k}, \Lambda, \|\mathcal{A}\|_{\mathcal{C}^{N+3}(\bar{\Gamma})}, \|f\|_{\mathcal{C}^{N+3}(\bar{\Gamma})}, \|\vec{\gamma}\|_{\mathcal{C}^{N+3}(\bar{\Gamma})}, \Omega, \Omega' \right)$$

such that

$$\sum_{|\vec{\alpha}| \leq N} \|D^{\vec{\alpha}} w\|_{L^\infty(\Omega')} \leq \mathcal{C} \|w\|_{L^\infty(\Omega')}, N.$$

Here $\tilde{\Gamma} = \Omega' \times \left[-2\|w\|_{L^\infty(\Omega')}, 2\|w\|_{L^\infty(\Omega')}\right]$ and B denotes $B_\gamma, B_A, B_{A'}$.

Remark 2.14. A special case of Theorem 2.13 is established in Theorem 2.4 in [10], namely when $n = 2$, $\vec{k}(x_1, x_2, z) = (1, k(x_1, z))$ is independent of x_2 with

$$(22) \quad |\partial_z k(x_1, z)| \leq C k(x_1, z)^{\frac{3}{2}},$$

and $f = 0$, $\vec{\gamma} = 0$. Also, much more is required there of the solution w , namely w must satisfy (see (2.22) in [10])

$$\begin{aligned} 1 + (\partial_{x_1} w(x_1, x_2))^2 &\leq C \partial_{x_2} w(x_1, x_2), \\ k(x_1, w(x_1, x_2)) \partial_{x_2} w(x_1, x_2) &\leq C. \end{aligned}$$

These restrictions are removed in Theorem 2.13, in which we also generalize the result to higher dimensions and allow lower order terms.

An important improvement found in Theorem 2.13 relative to Theorem 2.4 in [10] is the reduction of the power $3/2$ in (22) to the sharp power 1. See Section 3.

The next lemma shows that if only the first part of Condition 2.10 holds, then the bilinear form induced by $\mathcal{A}(x, z)$ is equivalent to one which is independent of z in any set on which z is bounded.

Lemma 2.15. Let $\mathcal{A} = (a_{ij}(x, z))_{i,j=1,\dots,n}$ be a smooth symmetric matrix satisfying (9) in Γ and such that for every $\Omega' \Subset \Omega$ and $M_0 \geq 1$, there exists $\tilde{B} = \tilde{B}_{\mathcal{A}}(M_0, \text{diam } \Omega, \text{dist}(\Omega', \partial\Omega))$ such that

$$(23) \quad |\partial_z \mathcal{A}(x, z) \xi|^2 \leq \tilde{B}^2 k^*(x, z) \xi^t \mathcal{A}(x, z) \xi, \quad (x, z) \in \Gamma'_{M_0}, \xi \in \mathbb{R}^n.$$

Then there exists $C = C(\tilde{B}, M_0)$ such that for all $(x, z), (x, \tilde{z}) \in \Gamma'_{M_0}$ and $\xi \in \mathbb{R}^n$,

$$C^{-1} \xi^t \mathcal{A}(x, z) \xi \leq \xi^t \mathcal{A}(x, \tilde{z}) \xi \leq C \xi^t \mathcal{A}(x, z) \xi.$$

Moreover, for all $i = 1, \dots, n$ and $(x, z), (x, \tilde{z}) \in \Gamma'_{M_0}$,

$$(24) \quad \tilde{C}^{-1} k^i(x, z) \leq k^i(x, \tilde{z}) \leq \tilde{C} k^i(x, z),$$

where $\tilde{C} = \tilde{C}(\mathcal{C}, \Lambda)$.

Proof. By (23), for all $\xi \in \mathbb{R}^n$, the function $h(x, z, \xi) = \xi^t \mathcal{A}(x, z) \xi$ satisfies

$$\begin{aligned} |\partial_z h(x, z, \xi)|^2 &= |\xi^t \mathcal{A}_z(x, z) \xi|^2 \\ &\leq |\xi|^2 |\mathcal{A}_z(x, z) \xi|^2 \\ &\leq |\xi|^2 \tilde{B}^2 k^*(x, z) \xi^t \mathcal{A}(x, z) \xi \\ &= |\xi|^2 \tilde{B}^2 k^*(x, z) h(x, z, \xi), \end{aligned}$$

for all $(x, z) \in \Gamma'_{M_0} = \Omega' \times [-M_0, M_0]$ and $\xi \in \mathbb{R}^n$. Since from (9) we have

$$k^*(x, z) |\xi|^2 = \sum_{i=1}^n k^i(x, z) \xi_i^2 \leq \sum_{i=1}^n k^i(x, z) \xi_i^2 \leq \xi^t \mathcal{A}(x, z) \xi = h(x, z, \xi),$$

we obtain

$$|\partial_z h(x, z, \xi)| \leq \tilde{B} h(x, z, \xi).$$

By Gronwall's inequality it follows that for some $C = C(\tilde{B}, M_0)$,

$$C^{-1} \xi^t \mathcal{A}(x, z) \xi \leq \xi^t \mathcal{A}(x, \tilde{z}) \xi \leq C \xi^t \mathcal{A}(x, z) \xi, \quad (x, z), (x, \tilde{z}) \in \Gamma'_{M_0}, \xi \in \mathbb{R}^n.$$

Inequality (24) then follows immediately from (9). \square

2.2. Hypoellipticity Main Results. Our main results Theorems 2.17 and 2.18 are obtained as applications of the a priori estimates above. Theorem 2.18 establishes smoothness of weak solutions, and Theorem 2.17 deals with existence, uniqueness, and regularity of strong solutions to the Dirichlet problem. The concept of weak solutions of our infinitely degenerate operators is similar to that of classical weak solutions for accretive operators, defined via the associated Hilbert space. We denote by $H_{\mathcal{X}}^{1,2}(\Omega)$ the Hilbert space on Ω induced by the quadratic form $\mathcal{X}(x, \xi) = \sum_{i=1}^n k^i(x, 0) \xi_i^2$ (see Appendix, Section 7.1.1). We say that w is a weak solution of (6) in Ω if $w \in H_{\mathcal{X}}^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$$\int \mathcal{A}(x, w) \nabla w \cdot \nabla \varphi \, dx - \int \varphi \vec{\gamma}(x, w) \cdot \nabla w \, dx - \int f(x, w) \varphi \, dx = 0$$

for all $\varphi \in Lip_0(\Omega)$. See Section 7 of the Appendix for a detailed discussion of the degenerate Sobolev spaces $H_{\mathcal{X}}^{1,2}(\Omega)$ and the meaning of ∇w if $w \in H_{\mathcal{X}}^{1,2}(\Omega)$.

Some extra conditions are required in order to make our approximation scheme work.

Definition 2.16 (Strongly convex domain). *A convex domain $\Phi \subset \mathbb{R}^n$ with $\partial\Phi \in \mathcal{C}^2$ is called strongly convex (with convex character λ_0) if there exists $\lambda_0 > 0$ such that*

$$\inf_{p \in \partial\Omega} \min_{1 \leq i \leq n-1} \lambda^i(p) = \lambda_0 > 0,$$

where $\lambda^1(p), \dots, \lambda^{n-1}(p)$ denote the principal curvatures (see [6] p.354) of $\partial\Phi$ at a point $p \in \partial\Phi$.

Theorem 2.18 is obtained from the following existence and uniqueness theorem for the Dirichlet problem.

Theorem 2.17 (Dirichlet problem). *Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a bounded open set and let $\Omega \Subset \tilde{\Omega}$ be a strongly convex domain. For $\Gamma = \tilde{\Omega} \times \mathbb{R}$, suppose $\vec{k} \in \mathcal{C}^2(\Gamma)$ and $\mathcal{A}(x, z), f(x, z), \vec{\gamma}(x, z) \in \mathcal{C}^\infty(\Gamma)$ are such that*

- (i) $\vec{k}(x, z)$ satisfies the nondegeneracy Condition 2.3 in Γ ,
- (ii) \mathcal{A} satisfies the diagonal Condition 2.5 in Γ ,
- (iii) $f(x, z) \text{ sign } z \leq 0$ and $f_z(x, z) \leq 0$ in Γ ,
- (iv) $\vec{\gamma}$ is of subunit type with respect to \mathcal{A} in Γ ,
- (v) $\vec{\gamma}$ has compact support in Ω in the x variable, locally in the z variable, i.e., for all $M_0 > 0$ there exists open $\Omega' \Subset \Omega$ such that $\vec{\gamma}(x, z) = 0$ if $(x, z) \in (\Omega \setminus \overline{\Omega'}) \times [-M_0, M_0]$,
- (vi) \mathcal{A} satisfies the super subordination Condition 2.10 in Γ ,
- (vii) $\vec{\gamma}$ satisfies the super subordination Condition 2.10 in Γ .

Then given any continuous function φ on $\partial\Omega$, there exists a strong solution w to the Dirichlet problem

$$(25) \quad \begin{cases} \operatorname{div} \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) &= 0 & \text{in } \Omega \\ w &= \varphi & \text{on } \partial\Omega, \end{cases}$$

i.e., there exists w which is continuous in $\overline{\Omega}$, equal to φ on $\partial\Omega$, and a strong solution of the differential equation in Ω . Moreover, $w \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, and for any nonnegative integer N and subdomain $\Omega' \Subset \Omega$, there exists a constant $\mathcal{C}_N =$

$$\mathcal{C}_N \left(n, B, \vec{k}, \Lambda, \|\varphi\|_{L^\infty(\partial\Omega)}, \|\mathcal{A}\|_{\mathcal{C}^{N+2}(\tilde{\Gamma})}, \|f\|_{\mathcal{C}^{N+1}(\tilde{\Gamma})}, \|\vec{\gamma}\|_{\mathcal{C}^{N+1}(\tilde{\Gamma})}, \lambda_0, \Omega, \Omega' \right)$$

such that $\sum_{|\vec{\alpha}| \leq N} \|D^{\vec{\alpha}} w\|_{L^\infty(\Omega')} \leq \mathcal{C}_N$. Here $\tilde{\Gamma} = \Omega \times [-2\|\varphi\|_{L^\infty(\partial\Omega)}, 2\|\varphi\|_{L^\infty(\partial\Omega)}]$, B denotes the various constants in Condition 2.10, and $\lambda_0 = \lambda_0(\mathcal{A}, \Omega)$ is the convex character of $\partial\Omega$.

Moreover, if $\vec{\gamma} \equiv 0$, then the solution w is unique.

An important consequence of Theorem 2.17 in the case $\vec{\gamma} \equiv 0$ is the following interior regularity result:

Theorem 2.18 (Regularity of solutions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\Gamma = \Omega \times \mathbb{R}$. Suppose that $\vec{k} \in C^2(\Gamma)$, that $\mathcal{A}(x, z), f(x, z) \in C^\infty(\Gamma)$ and satisfy (i), (ii), (iii), (vi) from Theorem 2.17, and that \mathcal{A} satisfies the super subordination Condition 2.10 in Γ . Then any weak solution w of*

$$\operatorname{div} \mathcal{A}(x, w) \nabla w + f(x, w) = 0 \quad \text{in } \Omega$$

which is continuous in Ω is also a strong solution and satisfies $w \in C^\infty(\Omega)$. Moreover, for any nonnegative integer N and subdomain $\Omega'' \Subset \Omega' \Subset \Omega$, there exists a constant

$$\mathcal{C}_N = \mathcal{C}_N \left(\|w\|_{L^\infty(\Omega')}, n, B, \vec{k}, \Lambda, \|\mathcal{A}\|_{C^{N+2}(\bar{\Gamma})}, \|f\|_{C^{N+1}(\bar{\Gamma})}, \Omega, \Omega', \Omega'' \right)$$

such that $\sum_{|\vec{\alpha}| \leq N} \|D^{\vec{\alpha}} w\|_{L^\infty(\Omega'')} \leq \mathcal{C}_N$. Here $\tilde{\Gamma} = \Omega' \times [-2\|w\|_{L^\infty(\Omega')}, 2\|w\|_{L^\infty(\Omega')}]$ and B denotes the relevant constants in Condition 2.10.

Note that in case $n = 2$, the super subordination Condition 2.10 reduces to (19) if $\mathcal{A} = \operatorname{diag}(1, k)$ and $\vec{\gamma} = 0$. Moreover, in this case, whether $\vec{\gamma} = 0$ or not, the nondegeneracy Condition 2.3 means that given $(x_1, x_2) \in \Omega$ and $\varepsilon > 0$, there exist $0 < r^1, r^2 < \varepsilon$ such that $k(x_1 \pm r^1, x_2, z) > 0$ if $|\xi_2 - x_2| \leq r^2$.

As an example in case $n \geq 2$, we consider a diagonal matrix

$$\mathcal{A}(x, z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & k^2(x, z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k^n(x, z) \end{pmatrix},$$

where k^i are smooth nonnegative functions satisfying the nondegeneracy Condition 2.3, and such that

$$|k_z^i(x, z)| \leq B k^*(x, z), \quad i = 1, \dots, n, \quad (x, z) \in \Gamma,$$

with $k^* = \min(k^1, \dots, k^n)$. Then \mathcal{A} satisfies the hypotheses of Theorem 2.18. In particular, if $k(x, z)$ is nonnegative and satisfies $|k_z| = O(k)$ as $k \rightarrow 0$, then $\mathcal{A} = \operatorname{diag}(1, k, \dots, k)$ is an admissible matrix for Theorem 2.18 provided property (ii) of the nondegeneracy Condition 2.3 holds.

Remark 2.19. *As a consequence of the previous observations in the case $\mathcal{A} = \operatorname{diag}(1, k^2, \dots, k^n)$, we partially recover Fedii's two-dimensional result [3] that $\partial_{x_1}^2 + k(x_1) \partial_{x_2}^2$ is hypoelliptic if k is smooth and positive for all $x_1 \neq 0$. Indeed, since $k(x_1)$ is independent of the z variable it automatically satisfies $|k_z| = O(k)$ as $k \rightarrow 0$. We only partially recover Fedii's result because our theorem applies only to continuous weak solutions.*

We also obtain a partial extension (namely, for *continuous* weak solutions) to higher dimensions of Fedii's result:

Theorem 2.20. *Let $k^i(x_1, \dots, x_n)$, $i = 2, \dots, n$, be smooth functions in \mathbb{R}^n such that k^i is independent of the i^{th} variable, i.e.,*

$$k^i(x_1, \dots, x_n) = k^i(\hat{x}^i), \quad \text{with } \hat{x}^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

and $k^i(x) > 0$ if $\hat{x}^i \neq 0$. Then any continuous weak solution of

$$\{\partial_{x_1}^2 + k^2(x) \partial_{x_2}^2 + \cdots + k^n(x) \partial_{x_n}^2\} w = 0$$

in \mathbb{R}^n is smooth everywhere.

Remark 2.21. *It is shown in [7] that if $k(x_1)$ is smooth and positive for all $x_1 \neq 0$, then $\partial_{x_1}^2 + k(x_1) \partial_{x_2}^2 + \partial_{x_3}^2$ is hypoelliptic in \mathbb{R}^3 if and only if $\lim_{x_1 \rightarrow 0} x_1 \log k(x_1) = 0$. We are not able to recover this result since $k(x_1)$ vanishes identically at all points of the form $(x_1, x_2, x_3) = (0, x_2, x_3)$, and so the hypothesis of our theorem is not met. Also, our solutions are required to be continuous. On the other hand, from Theorem 2.20, we see that if $k(x_1, x_3)$ is smooth and positive for all $(x_1, x_3) \neq (0, 0)$, then every weak solution of $\{\partial_{x_1}^2 + k(x_1, x_3) \partial_{x_2}^2 + \partial_{x_3}^2\} w = 0$ is smooth in the interior of the domain of continuity of w in \mathbb{R}^3 .*

The following is a striking consequence of Theorem 2.18 in \mathbb{R}^2 :

Theorem 2.22. *If $k(x_1, x_2, z)$ is smooth, nonnegative, satisfies*

$$(26) \quad |\partial_z k| = O(k),$$

and $k(\cdot, \cdot, 0)$ does not vanish identically on any horizontal line segment in Ω , then any continuous weak solution w of

$$(27) \quad \partial_{x_1}^2 w + \partial_{x_2} k(x_1, x_2, w(x_1, x_2)) \partial_{x_2} w = 0, \quad (x_1, x_2) \in \Omega,$$

is smooth in Ω .

Proof. We will prove that the hypotheses of Theorem 2.18 are satisfied by $\vec{k} = (1, k(x_1, x_2, z))$ and $\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. Since $k^1 \equiv 1$, property (i) in the nondegeneracy Condition 2.3 is trivially satisfied. Since $k(\cdot, \cdot, 0)$ is nonnegative, continuous and does not vanish identically on any horizontal segment, given $\varepsilon > 0$ and $(x_1, x_2) \in \Omega$, there exist $\bar{x}_1 < x_1 < \bar{x}'_1$ with $|x_1 - \bar{x}_1| = |x_1 - \bar{x}'_1| = r^1 < \varepsilon$, such that $k(\bar{x}_1, x_2, 0) > 0$ and $k(\bar{x}'_1, x_2, 0) > 0$. From (26) and Lemma 2.11 it follows that $k(x_1, x_2, z)$ is either identically zero as a function of z or strictly positive in z , and hence $k(\bar{x}_1, x_2, z) > 0$ and $k(\bar{x}'_1, x_2, z') > 0$. Then property (ii) in the nondegeneracy Condition 2.3 follows from the continuity of k with respect to the second variable and therefore \vec{k} satisfies hypothesis (i) in Theorem 2.18. Since (26) holds, \mathcal{A} satisfies (16). Since \mathcal{A} is diagonal, (17) then follows from Remark 2.12. Thus \mathcal{A} is super subordinate, so hypotheses (ii) and (iv) in Theorem 2.18 are satisfied. Since $f = 0$, hypothesis (iii) of Theorem 2.18 is trivially satisfied. \square

From Theorem 2.22 and the techniques used in [10], we can derive an extension of Theorem 2.1 in [10], which is a regularity result for convex solutions to Monge-Ampère equations. Consider a smooth, bounded, strongly convex domain $\Phi \subset \mathbb{R}^2$. Given a convex function $u \in \mathcal{C}^1(\Phi)$, following [10] we set

$$\omega_-(s) = \omega_-(s, \Phi, u) = \inf_{t: (s, t) \in \Phi} u_t(s, t) \quad \text{and} \quad \omega_+(s) = \omega_+(s, \Phi, u) = \sup_{t: (s, t) \in \Phi} u_t(s, t)$$

for any s lying in the projection of Φ onto the s -axis. Let

$$\mathbf{I}_s = \{\omega : \omega_-(s) < \omega < \omega_+(s)\} \quad \text{if } \omega_-(s) < \omega_+(s) \text{ and } \mathbf{I}_s = \emptyset \text{ otherwise, and}$$

$$\Phi_u = \{(s, t) \in \Phi : u_t(s, t) \in \mathbf{I}_s\}.$$

Note that

- $u(s, t)$ is affine in the t variable if and only if $\mathbf{I}_s = \emptyset$.
- If $u \in \mathcal{C}^1(\Phi)$ is not affine in the t variable for any fixed s , then Φ_u is an open connected set. Indeed, to see that Φ_u is open, let $(s, t) \in \Phi_u$. Then $\omega_-(s) < u_t(s, t) < \omega_+(s)$, and since $u \in \mathcal{C}^1(\Phi)$, the functions ω_- , ω_+ and u_t are continuous. Hence, for (s', t') near (s, t) , $\omega_-(s') < u_t(s', t') < \omega_+(s')$. Therefore Φ_u is open. To see that Φ_u is connected it suffices to show it is pathwise connected. This follows from the fact that the midpoint $(\omega_-(s) + \omega_+(s))/2$ of \mathbf{I}_s is a continuous function of s . Note the arguments above only use the continuity of u_t . See [10] for further discussion about when Φ_u is connected.
- Even if u is not affine in the t variable for any fixed s , the set

$$\Pi_u = \{(s, t) \in \Phi_u : u_t(s, t) = \omega_-(s) \text{ or } u_t(s, t) = \omega_+(s)\}$$

may be non-empty. This exceptional set Π_u is composed by what are called “Pogorelov segments” in [10].

Theorem 2.23. *Suppose $k(s, t)$ is smooth, nonnegative, satisfies $|\partial_t k| = O(k)$, and $k(\cdot, 0)$ does not vanish identically on any horizontal line segment in Φ , where Φ is as above. If $u \in \mathcal{C}^1(\Phi)$ is a convex solution of the Monge-Ampère boundary value problem*

$$\begin{aligned} \det \begin{pmatrix} u_{ss} & u_{st} \\ u_{ts} & u_{tt} \end{pmatrix} &= k(s, t), & (s, t) \in \Phi, \\ u &= \phi(s, t), & (s, t) \in \partial\Phi, \end{aligned}$$

where ϕ is smooth on $\partial\Phi$, then u is smooth in Φ_u .

To obtain our main results, we follow the approach in [10] for two-dimensional equations, although our objectives are more general. We consider equations in any dimension at least two, our equations may include a first order drift term and a zero order term, and our notion of solution is more general since we only require continuity instead of Lipschitz continuity. To prove our main hypoellipticity result, Theorem 2.18, we use an approximation argument based on the a priori estimates in Theorem 2.9 and on the construction in Lemma 7.6 of new custom-built barriers. One of the core ingredients needed to derive Theorem 2.9 is the interpolation inequality given in Lemma 4.7, proved in [9].

3. SHARPNESS

Our results are sharp in the sense that the power 1 in the super subordination Condition 2.10 cannot be decreased. Indeed, we will now show that for any $\varepsilon > 0$, there exists a nonnegative smooth function $k = k(x, y, z)$ in $\mathbb{R}^2 \times \mathbb{R}$ which is not identically zero on any horizontal segment in \mathbb{R}^2 (moreover $k > 0$ unless $x = z = 0$) and satisfies

$$|\partial_z k(x, y, z)| \leq C [k(x, y, z)]^{1-\varepsilon} \quad \text{in } \Omega \times \mathbb{R},$$

and that there is a continuous weak solution w of

$$(28) \quad \partial_x^2 w + \partial_y k(x, y, w(x, y)) \partial_y w = 0$$

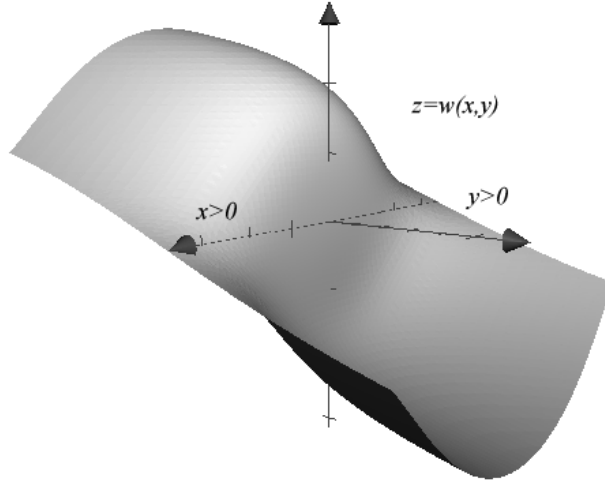
in any neighborhood Ω of the origin, *but w is not smooth in Ω* . This example is derived by applying the partial Legendre transform to non-smooth solutions of the Monge-Ampère equation which are suitable powers of the distance function to the origin.

Given $\varepsilon > 0$, let m be a positive integer such that

$$(29) \quad \frac{1}{4m-2} < \varepsilon.$$

Consider the function $w = w(x, y)$ defined implicitly by the equation

$$(30) \quad F(x, y, w) = 0 \quad \text{where } F(x, y, z) = z(|x|^2 + |z|^2)^{m-\frac{1}{2}} + y.$$



Since $F(0, 0, 0) = 0$ and $F_z(x, y, z) = (|x|^2 + |z|^2)^{m-\frac{3}{2}}(x^2 + 2mz^2)$, it follows from the implicit function theorem that $w = w(x, y)$ is well-defined by (30) and smooth in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Also, w extends continuously to $(x, y) = (0, 0)$ with $w(0, 0) = 0$, and thus for any neighborhood Ω of the origin, $w \in C^0(\overline{\Omega}) \cap C^\infty(\Omega \setminus \{(0, 0)\})$. Moreover, $w(x, y) = 0$ if and only if $y = 0$.

Now let

$$f(x, y) = x(|x|^2 + |w(x, y)|^2)^{m-\frac{1}{2}} \quad (x, y) \in \Omega.$$

Then $f(0, y) = 0$, $f(x, 0) = x|x|^{2m-1}$ and $f(x, y) = -\frac{xy}{w(x, y)}$ if $w(x, y) \neq 0$. By direct computation, if $(x, y) \neq (0, 0)$,

$$(31) \quad \begin{cases} f_x(x, y) &= -2m(x^2 + w^2)^{2m-1} w_y(x, y) \\ f_y(x, y) &= w_x(x, y). \end{cases}$$

Thus, if $k(x, y, z) = k(x, z)$ is the smooth function in $\mathbb{R}^2 \times \mathbb{R}$ defined by

$$k(x, z) = 2m(x^2 + z^2)^{2m-1},$$

then by using the formulas $F_x = (2m-1)xz(|x|^2 + |z|^2)^{m-\frac{3}{2}}$ and $F_y = 1$, we have

$$\begin{aligned} |w_x|^2 &= \left| -\frac{(2m-1)xw}{x^2 + 2mw^2} \right|^2 \leq \left| \frac{(2m-1)}{2} \frac{x^2 + w^2}{x^2 + 2mw^2} \right|^2 \leq m^2, \\ k|w_y|^2 &= \frac{2m(x^2 + w^2)^{2m-1}}{(x^2 + w^2)^{2m-3}(x^2 + 2mw^2)^2} \\ &= 2m \frac{(x^2 + w^2)^2}{(x^2 + 2mw^2)^2} \leq 2m. \end{aligned}$$

In particular, this implies that

$$\int_{\Omega} (|w_x|^2 + k|w_y|^2) dx dy \leq 3m^2 |\Omega|.$$

that is, $w \in H_{\mathcal{X}}^{1,2}(\Omega) \cap L^\infty(\Omega)$, where $\mathcal{X}(x, y, w, \xi_1, \xi_2) = \xi_1^2 + k\xi_2^2$. From (31) it follows that $w(x, y)$ is a continuous weak solution of the quasilinear equation (28). Moreover, as a function of (x, y) , $k(x, y, z)$ does not vanish on any horizontal line segment, and it satisfies

$$(32) \quad \begin{aligned} |\partial_z k(x, z)| &= 4m(2m-1)|z|(x^2 + z^2)^{2m-2} \\ &\leq C_m(|x|^2 + |z|^2)^{2m-\frac{3}{2}} = C_m k(x, z)^{1-\frac{1}{4m-2}} \leq C_m k^{1-\varepsilon}, \end{aligned}$$

where we used the inequality $|z| \leq \sqrt{x^2 + z^2}$ and the bound (29) for m .

On the other hand, from (30), noting that y and $w(x, y)$ have the same sign, we have

$$w(0, y) = (\text{sign } y) |y|^{\frac{1}{2m}}.$$

Hence $w(x, y) \notin \mathcal{C}^{\frac{1}{2m}+\delta}(\Omega)$ for any $\delta > 0$. In particular, w is not smooth in Ω .

4. PRELIMINARIES

4.1. Notation. Throughout the paper, C will denote a constant that may change from line to line but that is independent of any significant quantities. In general, C may depend on the dimension n , \vec{k} , and the fixed cutoff functions defined below. We will use calligraphic \mathcal{C} to denote a function of one or more variables, increasing in each variable separately, that may also change from line to line, but that is independent of any significant parameters except its variables.

We denote by $\|f\|_{L^\infty(\Omega)}$ the essential supremum of $|f|$ in Ω and by $\|f\|_{L^p(\Omega)}$ the L^p -norm of f in Ω :

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

When $\Omega = \mathbb{R}^n$ we omit mentioning the set. The collection of real-valued functions in Ω with m continuous (but not necessarily bounded) derivatives in Ω will be denoted $\mathcal{C}^m(\Omega)$, and for $f \in \mathcal{C}^m(\Omega)$ we let

$$(33) \quad \|f\|_{\mathcal{C}^m(\Omega)} = \sum_{i=0}^m \sum_{|\vec{\alpha}|=i} \|\partial^{\vec{\alpha}} f\|_{L^\infty(\Omega)},$$

where $\partial^{\vec{\alpha}} = (\partial_1)^{\alpha_1} \cdots (\partial_n)^{\alpha_n}$ for $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$.

Let $\Gamma'_{M_0} \subset \Gamma$ be the domains in $\mathbb{R}^n \times \mathbb{R}$ given in (13) and let $k^i(x, z) \in \mathcal{C}^2(\Gamma)$ be nonnegative, $i = 1, \dots, n$. From now on, let $\vec{k}(x, z) = (k^1(x, z), \dots, k^n(x, z))$ satisfy the nondegeneracy Condition 2.3 with

$$(34) \quad k^1(x, z) = 1 \quad \text{for all } (x, z) \in \Gamma,$$

which clearly implies property (i) in Condition 2.3. Since our theorems are local, this assumption causes no loss of generality.

4.2. Boxes around points. Given $x \in \Omega$, we will consider rectangular neighborhoods \mathcal{R} of x of the form described in Section 2.1, with $x \in \frac{1}{3}\mathcal{R}$. The maximum sidelength R of \mathcal{R} will be chosen so that $2\mathcal{R} \subset \Omega$ and possibly even smaller to allow the absorption of various terms involving R as a factor. We will always assume that \mathcal{R} satisfies property (ii) of the nondegeneracy Condition 2.3. Since \vec{k} is continuous, given such \mathcal{R} and $M_0 > 0$, there exist positive numbers $\delta^i = \delta^i(\vec{k}, \mathcal{R}, M_0)$ such that $\delta^i < \frac{1}{2}r^i$, where r^i denotes the i^{th} half-sidelength of \mathcal{R} , $i = 1, \dots, n$, and

$$(35) \quad \begin{aligned} k^i(y, z) &> 0 \quad \text{whenever } z \in [-M_0, M_0] \text{ and} \\ y &\in \mathcal{T}_i(\mathcal{R}, \delta^i) = \{Y \in \mathbb{R}^n : \text{dist}(Y, \mathcal{T}_i(\mathcal{R})) \leq \delta^i\}. \end{aligned}$$

Remark 4.1. Under the hypotheses of Theorem 2.18, or more precisely, if \mathcal{A} satisfies (16) and (9), the parameters δ^i above can be taken independent of M_0 . Indeed, (16) and Lemma 2.15 imply that for fixed $x, \xi \in \mathbb{R}^n$, the function $\xi^t \mathcal{A}(x, z) \xi$ is either identically zero in z or strictly positive in z . Therefore, (9) implies a similar property for each $k^i(x, z)$.

4.3. A class of adapted cutoff functions. A cutoff function is any nonnegative smooth function with compact support, i.e., φ is a cutoff function if $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\varphi \geq 0$.

We now define a special class of cutoff functions around x which are adapted to our operator as in [11] (see also [3]). The main property of these functions is that they are supported in a (small enough) neighborhood of x , while their derivatives are supported *away* from x , essentially where \vec{k} has positive components.

Definition 4.2 (Supporting relation). *Given two cutoff functions $\zeta, \xi \in \mathcal{C}_0^\infty(\Omega)$, we say that ζ supports ξ and denote it $\xi \succeq \zeta$ or $\zeta \preceq \xi$ if $\xi = 1$ on a neighborhood of support(ζ). Note in particular that if $\xi \succeq \zeta$ then $\zeta \xi = \zeta$ and $\|\zeta\|_{L^\infty} \xi \geq \zeta$.*

Definition 4.3 (Special cutoff functions). *Let $x \in \Omega$, $M_0 \geq 1$, $\mathcal{R} = \tilde{x} + ([-r^1, r^1] \times \dots \times [-r^n, r^n])$ be a rectangular box with $x \in \frac{1}{3}\mathcal{R}$, and $\delta^i = \delta^i(\vec{k}, \mathcal{R}, M_0) > 0$ be as in (35). Let*

$$\eta_i, \phi_i, \zeta_i, \theta_i \in \mathcal{C}_0^\infty(\tilde{x}_i + (-2r^i, 2r^i)), \quad 1 \leq i \leq n,$$

be functions of one variable which satisfy the following:

- (i) $0 \leq \eta_i, \phi_i, \zeta_i, \theta_i \leq 1$
- (ii) η_i, ϕ_i and ζ_i are equal to 1 in $\tilde{x}_i + [- (r^i - \delta^i), r^i - \delta^i]$ and vanish outside $\tilde{x}_i + [- (r^i + \delta^i), r^i + \delta^i]$; and for all integers $m \geq 1$ and some universal constants $c_m \geq 1$,

$$(36) \quad \left(\frac{1}{c_m \delta^i} \right)^m \leq \left\| \frac{d^m}{dt^m} \eta_i \right\|_{L^\infty(\mathbb{R})} + \left\| \frac{d^m}{dt^m} \phi_i \right\|_{L^\infty(\mathbb{R})} + \left\| \frac{d^m}{dt^m} \zeta_i \right\|_{L^\infty(\mathbb{R})} \leq \left(\frac{c_m}{\delta^i} \right)^m.$$

- (iii) $\eta_i \preceq \phi_i \preceq \zeta_i$ and $\eta_i \leq \phi_i \leq \zeta_i$
- (iv) Let J^i be the smallest set of the form $J^i = \tilde{x}_i + ([-b^i, -a^i] \cup [a^i, b^i]) \subset \mathbb{R}$, with $0 < a^i < b^i$, such that

$$\text{support}(\eta'_i) \bigcup \text{support}(\phi'_i) \bigcup \text{support}(\zeta'_i) \subset J^i.$$

Note by (ii) that $J^i \subset \tilde{x}_i + ([-r^i - \delta^i, -r^i + \delta^i] \cup [r^i - \delta^i, r^i + \delta^i])$. Let $\theta_i \equiv 1$ on J^i , so that $\theta_i \equiv 1$ on the supports of η'_i, ϕ'_i and ζ'_i , i.e., $\eta'_i \preceq \theta_i, \phi'_i \preceq \theta_i$, and $\zeta'_i \preceq \theta_i$.

- (v) $\text{support}(\theta_i) \subset \tilde{x}_i + ([-r^i - 2\delta^i, -r^i + 2\delta^i] \cup [r^i - 2\delta^i, r^i + 2\delta^i])$; in particular, $\tilde{x}_i \notin \text{support}(\theta_i)$. Moreover, we assume that for all integers $m \geq 1$ and for some universal constants $C_m > 0$,

$$(37) \quad \left(\frac{1}{C_m \delta^i} \right)^m \leq \left\| \frac{d^m}{dt^m} \theta_i \right\|_{L^\infty(\mathbb{R})} \leq \left(\frac{C_m}{\delta^i} \right)^m.$$

- (vi) $|\eta'_i|$, $|\phi'_i|$ and $|\zeta'_i|$ are smooth functions (see the discussion following the figure below).

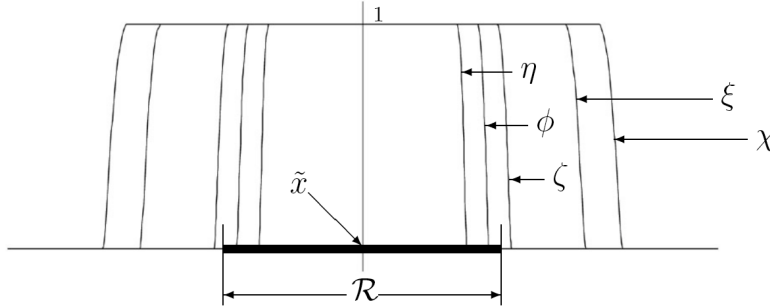
Finally, let

$$\begin{aligned} \eta(x) &= \prod_{i=1}^n \eta_i(x_i), & \phi(x) &= \prod_{i=1}^n \phi_i(x_i), \\ \zeta(x) &= \prod_{i=1}^n \zeta_i(x_i), & \varrho_i(x) &= \theta_i(x_i) \prod_{j \neq i} \zeta_j(x_j), \end{aligned}$$

and let $\xi, \chi \in \mathcal{C}_0^\infty(2\mathcal{R})$ satisfy

$$\begin{aligned} \xi &= \chi = 1 & \text{in } \mathcal{R} \\ \zeta &\preceq \xi \preceq \chi, & \zeta \leq \xi \leq \chi, \quad \text{and} \\ \varrho_i &\preceq \xi, & i = 1, \dots, n. \end{aligned}$$

The figure below will serve as a reminder of the order between some of the cutoff functions:



Property (vi) above is easily satisfied by assuming (in addition to properties (i) to (iv)) that η_i , ϕ_i and ζ_i are smooth, non-decreasing in $(-\infty, \tilde{x}_i)$ and non-increasing in (\tilde{x}_i, ∞) . Indeed, under such conditions and since these functions are constant on a neighborhood of \tilde{x}_i , their derivatives are of the form $\eta'_i = (\eta'_i)^+ - (\eta'_i)^-$, $\phi'_i = (\phi'_i)^+ - (\phi'_i)^-$ and $\zeta'_i = (\zeta'_i)^+ - (\zeta'_i)^-$ where all these functions are compactly supported, $(\eta'_i)^+$, $(\phi'_i)^+$ and $(\zeta'_i)^+$ are smooth, supported in $(-\infty, \tilde{x}_i)$ and nonnegative, while $(\eta'_i)^-$, $(\phi'_i)^-$ and $(\zeta'_i)^-$ are smooth, supported in (\tilde{x}_i, ∞) and nonnegative. It follows that $|\eta'_i| = (\eta'_i)^+ + (\eta'_i)^-$, $|\phi'_i| = (\phi'_i)^+ + (\phi'_i)^-$ and $|\zeta'_i| = (\zeta'_i)^+ + (\zeta'_i)^-$ are smooth functions.

It will be convenient to set

$$(38) \quad \begin{aligned} A^6 &= 1 + \|\nabla \eta\|_{L^\infty}^6 + \|\nabla \phi\|_{L^\infty}^6 + \|\nabla \zeta\|_{L^\infty}^6 + \|\nabla \varrho_1\|_{L^\infty}^6 + \cdots + \|\nabla \varrho_n\|_{L^\infty}^6 \\ &\quad + \|\nabla^2 \eta\|_{L^\infty}^3 + \|\nabla^2 \phi\|_{L^\infty}^3 + \|\nabla^2 \zeta\|_{L^\infty}^3 \\ &\quad + \|\nabla^3 \eta\|_{L^\infty}^2 + \|\nabla^3 \phi\|_{L^\infty}^2 + \|\nabla^3 \zeta\|_{L^\infty}^2 \end{aligned}$$

in order to collect constants in front of the lower order terms in what follows.

Remark 4.4. Note that A depends only on \vec{k} , \mathcal{R} and M_0 . Let $\delta^* = \delta^*(\vec{k}, \mathcal{R}, M_0) = \min_{i=1, \dots, n} \delta^i$ where $\delta^i = \delta^i(\vec{k}, \mathcal{R}, M_0)$ are as in (35). Then from (38), (36) and (37),

$$A \approx (\delta^*)^{-1}.$$

The main property of the cutoff functions η , ϕ and ζ above is that their i^{th} partial derivative, $i = 1, \dots, n$, is supported in the set $\mathcal{K}_i = \bigcap_{\ell \neq i, |z| \leq M_0} \{x : k^\ell(x, z) > 0\}$. Indeed, let $\sigma(x) =$

$\prod_{\ell=1}^n \sigma_\ell(x_\ell)$ with $\eta_\ell \leq \sigma_\ell \leq \zeta_\ell$, and fix $z \in [-M_0, M_0]$. It follows from Definition 4.3 (iv) and (v) that $\sigma'_\ell \preceq \theta_\ell$, i.e., $\text{supp}(\sigma'_\ell) \subset \{\theta_\ell = 1\}$. Hence,

$$\text{supp}(\sigma'_i) \subset \text{supp}(\theta_i) \subset \tilde{x}_i + \left([-r^i - 2\delta^i, -r^i + 2\delta^i] \cup [r^i - 2\delta^i, r^i + 2\delta^i] \right).$$

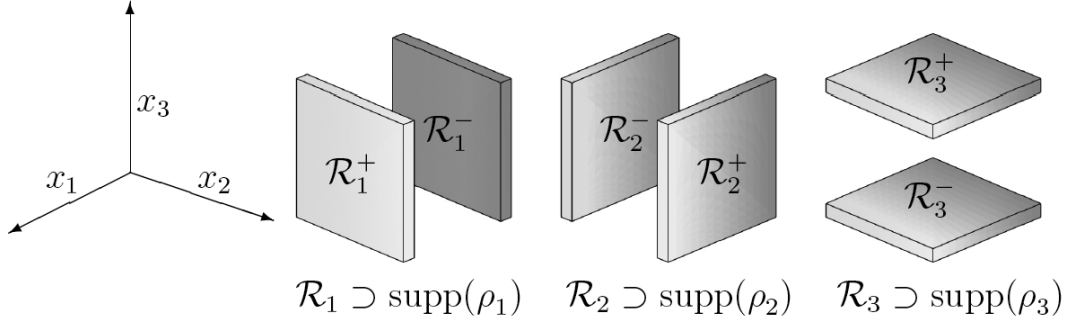
Hence, since $\text{supp}(\sigma_\ell) \subset \tilde{x}_\ell + [-r^\ell - \delta^\ell, r^\ell + \delta^\ell]$, we have $\text{supp}(\partial_i \sigma) \subset \text{supp}(\varrho_i)$. Now set $I_\ell = [-r^\ell - \delta^\ell, r^\ell + \delta^\ell]$ and $\mathcal{R}_i = \mathcal{R}_i^+ \cup \mathcal{R}_i^-$, where

$$(39) \quad \begin{aligned} \mathcal{R}_i^+ &= \tilde{x} + \left\{ \left(\prod_{\ell < i} I_\ell \right) \times [r^i - 2\delta^i, r^i + 2\delta^i] \times \left(\prod_{\ell > i} I_\ell \right) \right\}, \\ \mathcal{R}_i^- &= \tilde{x} + \left\{ \left(\prod_{\ell < i} I_\ell \right) \times [-(r^i + 2\delta^i), -(r^i - 2\delta^i)] \times \left(\prod_{\ell > i} I_\ell \right) \right\}. \end{aligned}$$

From (35) it follows that

$$\text{supp}(\partial_i \sigma) \subset \text{supp}(\varrho_i) \subset \mathcal{R}_i \subset \mathcal{K}_i,$$

as wanted. The figure below represents the sets \mathcal{R}_i when $n = 3$.



Since $\text{supp}(\varrho_i)$ is compact and \mathcal{K}_i is open, it follows from (24) that there exists $\tilde{C}_1 = \tilde{C}_1(\vec{k}, \mathcal{R}, M_0) > 0$ such that

$$\varrho_i(x) \min_{\ell \neq i} k^\ell(x, z) \geq \frac{1}{\tilde{C}_1} \varrho_i(x), \quad i = 1, \dots, n, \quad x \in \Omega \quad z \in [-M_0, M_0].$$

Since \vec{k} is bounded in any compact set, it follows from Condition 2.3 that there exists $C_1 = C_1(\vec{k}, \mathcal{R}, M_0)$ such that for $1 \leq i \leq n$,

$$(40) \quad \varrho_i(x) k^i(x, z) \leq C_1 \varrho_i(x) \min_{1 \leq j \leq n} k^j(x, z), \quad (x, z) \in \Omega \times [-M_0, M_0].$$

We will often want to show that a certain term is small by applying the one-dimensional Sobolev inequality in the x_1 -variable, i.e., by applying the estimate

$$(41) \quad \|\varphi\|_{L^2(\mathbb{R}^n)} \leq Cr^1 \|\partial_1 \varphi\|_{L^2(\mathbb{R}^n)},$$

where φ is a function with compact support in $2\mathcal{R}$ and C is a universal constant. Then by (9), (34), and the definition in (44) below, we have

$$(42) \quad \|\varphi\|_{L^2(\mathbb{R}^n)} \leq Cr^1 \|\nabla_{A,w} \varphi\|_{L^2(\mathbb{R}^n)} \quad \text{if } \text{support}(\varphi) \subset 2\mathcal{R}.$$

The constant factor r^1 which appears here will often be chosen small to help in absorption arguments, but it is important to observe that since $A \geq (\delta^1)^{-1} \geq 3(r^1)^{-1}$, we must ensure that a term to be shown small because it contains an r^1 factor *does not also include a factor of positive powers of A* .

For simplicity, we will often restrict our calculations to the case when the center \tilde{x} of \mathcal{R} is the origin.

4.4. Auxiliary Results. Given a weak solution $w \in H_{\chi}^{1,2}(\Omega) \cap L_{\infty}(\Omega)$ to (6), we denote $\mathbf{A}(x) = \mathcal{A}(x, w(x))$. It is convenient to define the *linear* operator

$$(43) \quad \mathcal{L}_w = \operatorname{div} \mathbf{A}(x) \nabla = \operatorname{div} \mathcal{A}(x, w(x)) \nabla, \quad x \in \Omega.$$

Given $\varphi \in H_{\chi}^{1,2}(\Omega)$, we denote by $\nabla_{\sqrt{\mathcal{A}}, w} \varphi$ the $\sqrt{\mathcal{A}}$ -gradient of φ , formally defined by

$$(44) \quad \nabla_{\sqrt{\mathcal{A}}, w} \varphi = \sqrt{\mathcal{A}(x, w(x))} \nabla \varphi.$$

See the Appendix for a discussion of the meaning of $\nabla \varphi$ in case φ is not smooth.

We now list four useful lemmas obtained in [9].

Lemma 4.5. *Let $u \in C^{\infty}(2\mathcal{R})$, ψ be a nonnegative cutoff function supported in $2\mathcal{R}$, and $\beta \in \mathbb{N}$. Then*

$$\begin{aligned} \int_{2\mathcal{R}} \left| \psi \nabla_{\sqrt{\mathcal{A}}, w} u^{\beta} \right|^2 &\leq \frac{2\beta^2}{2\beta-1} \left| \int_{2\mathcal{R}} (\psi \mathcal{L}_w u) (\psi u^{2\beta-1}) \right| \\ &\quad + \left(\frac{4\beta}{2\beta-1} \right)^2 \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \psi \right|^2 |u^{\beta}|^2, \end{aligned}$$

where \mathcal{L}_w is the linear operator (43), and $\nabla_{\sqrt{\mathcal{A}}, w}$ is given by (44).

Lemma 4.6. *Let \vec{k} satisfy Condition 2.3 and \mathcal{R} be a box satisfying property (ii) in Condition 2.3. For any smooth function φ and smooth cutoff function ψ of the form $\psi = \prod_{i=1}^n \psi_i(x_i)$, where $\eta_i \leq \psi_i \leq \zeta_i$ for all i (see Definition (4.3)), we have*

$$\left| \nabla_{\sqrt{\mathcal{A}}, w} \psi \right|^2 |\nabla \varphi|^2 \leq C_1 \Lambda |\nabla \psi|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} \varphi \right|^2,$$

with $\Lambda, C_1 = C_1(\vec{k}, \mathcal{R}, M_0)$ and $\nabla_{\sqrt{\mathcal{A}}, w}$ as in (9), (40) and (44).

Lemma 4.7. *Suppose that ζ, χ are cutoff functions as in Definition 4.3, and \mathcal{R} is a rectangle with $2\mathcal{R} \subset \Omega$ which satisfies property (i) of Condition 2.3. Then for each $q > n$, there exists $1 < p < 2$ such that for all $u \in C_0^1(2\mathcal{R})$ satisfying $u \preceq \chi$ (see Definition 4.2), all $\beta \in \mathbb{N}$, and all $0 < \epsilon \leq 1$,*

$$\int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 |u^{\beta}|^2 \leq \epsilon^{-1} \mathcal{C}(n, A, q, \mathcal{K}, \mathcal{R}, C_1, \Lambda) \|u^{\beta}\|_{L^p}^2 + \epsilon \int_{2\mathcal{R}} \left| \zeta \nabla_{\sqrt{\mathcal{A}}, w} u^{\beta} \right|^2,$$

where $\mathcal{K} = \|\nabla \chi \mathbf{A}\|_{L^q}$, and C_1 is as in Lemma 4.6.

Lemma 4.8. *Suppose that w is a smooth solution of (6) in $2\mathcal{R} \subset \Omega'$. Then for $\beta \in \mathbb{N}$ and any $0 < \alpha \leq 1$,*

$$\begin{aligned} &\sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\mathcal{L}_w w_{ij}) w_{ij}^{2\beta-1} \right| \\ &\leq 2 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} (\nabla w_j) \cdot \mathbf{A}_i \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + 2 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i (\nabla w_j) \cdot \mathbf{A}_z \nabla \zeta^2 w_{ij}^{2\beta-1} \right| \\ &\quad + \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_{ij} (\nabla w) \cdot \mathbf{A}_z \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + \frac{\mathcal{C}_0}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla^2 w|^{2\beta} \\ &\quad + \alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\vec{\gamma} \cdot \nabla w_{ij}^{\beta}|^2 + \alpha \int_{2\mathcal{R}} \zeta^2 |\vec{\gamma}_z \cdot \nabla w|^2 |\nabla^2 w|^{2\beta} \\ &\quad + 2 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 ((\partial_j \vec{\gamma}) \cdot \nabla w_i) w_{ij}^{2\beta-1} \right| + 2 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i (\nabla w) \cdot \mathbf{A}_{jz} \nabla \zeta^2 w_{ij}^{2\beta-1} \right| \\ &\quad + \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i w_j (\nabla w) \cdot \mathbf{A}_{zz} \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + \mathcal{C}_0 \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{6\beta} + \mathcal{C}_0, \end{aligned}$$

where $\vec{\gamma} = \vec{\gamma}(x, w)$, $f = f(x, w)$, $\mathbf{A} = \mathbf{A}(x, w)$, $\mathbf{A}_i = \mathbf{A}_i(x, w)$, etc., and

$$\mathcal{C}_0 = C \left\{ \|\mathcal{A}\|_{\mathcal{C}^3(\bar{\Gamma})} + \|f\|_{\mathcal{C}^2(\bar{\Gamma})} + \|\vec{\gamma}\|_{\mathcal{C}^2(\bar{\Gamma})} + 1 \right\}.$$

Remark 4.9. Lemma 4.8 is a slightly different version of Lemma 5.6 in [9] and readily follows from its proof. Indeed the fourth term on the right side of the conclusion of the original lemma is replaced, via straightforward changes in the proof, by the sixth and seventh terms on the right side above.

The following result is used in the proof of Theorem 2.17.

Lemma 4.10. Given u continuous on $[0, 1]$, there exists $w \in \mathcal{C}^0([0, 1]) \cap \mathcal{C}^2((0, 1])$ such that w is concave, strictly increasing, $w(x) \geq u(x)$ for all $x \in [0, 1]$, and $w(0) = u(0)$.

Proof. Let $\tilde{u}(x) = \max_{t \in [0, x]} u(t)$ for $x \in [0, 1]$, and $\tilde{u}(x) = \max_{t \in [0, 1]} u(t)$ for $x > 1$. Since $\max_{t \in [0, x]} u(t)$ is nondecreasing, $\tilde{u}(x)$ is nondecreasing in $[0, \infty)$, $\tilde{u}(x) \geq u(x)$ in $[0, 1]$, and $\tilde{u}(0) = u(0)$.

Next, for a smooth nonnegative function η with compact support in $[-1, 1]$ and $\int \eta = 1$, set $\eta_\varepsilon(x) = \frac{1}{\varepsilon} \eta\left(\frac{x}{\varepsilon}\right)$ and let $v(x) = \tilde{u} * \eta_{\frac{x}{2}}(2x)$ for $x > 0$, and $v(0) = u(0)$. Then $v(x) \geq \tilde{u}(x)$ for all x , and $v \in \mathcal{C}^0([0, \infty)) \cap \mathcal{C}^\infty((0, \infty))$.

Taking now $\tilde{v}(x) = \max_{t \in [0, x]} v(t) + x$, we have that $\tilde{v} \in \mathcal{C}^0([0, \infty)) \cap \mathcal{C}^\infty((0, \infty))$, $\tilde{v}(x) \geq v(x) \geq \tilde{u}(x)$ for all x , $\tilde{v}(0) = u(0)$, and \tilde{v} is strictly increasing.

By the fundamental theorem of calculus,

$$\tilde{v}(x) = \tilde{v}(1) - \int_x^1 \tilde{v}'(t) dt = \tilde{v}(1) - (1-x)\tilde{v}'(1) + \int_x^1 \int_t^1 \tilde{v}''(s) ds dt$$

for all $x \in (0, \infty)$. Let $[\tilde{v}''(s)]^+ = \max\{\tilde{v}''(s), 0\}$ and $[\tilde{v}''(s)]^- = \max\{-\tilde{v}''(s), 0\}$. Then $[\tilde{v}''(s)]^\pm$ are continuous in $(0, \infty)$ and $\tilde{v}''(s) = [\tilde{v}''(s)]^+ - [\tilde{v}''(s)]^-$. Since

$$\max_{x \in [0, 1]} \left| \int_x^1 \int_t^1 \tilde{v}''(s) ds dt \right| = \max_{x \in [0, 1]} |\tilde{v}(x) - \tilde{v}(1) + (1-x)\tilde{v}'(1)| < \infty,$$

it follows that the functions w_+ and w_- defined by

$$w_+(x) = \int_x^1 \int_t^1 [\tilde{v}''(s)]^+ ds dt \quad \text{and} \quad w_-(x) = \int_x^1 \int_t^1 [\tilde{v}''(s)]^- ds dt$$

are finite and belong to $\mathcal{C}^2(0, 1]$. Also note that $\tilde{v}(x) = \tilde{v}(1) - (1-x)\tilde{v}'(1) + w_+(x) - w_-(x)$. Moreover, since

$$(w_\pm(x))'' = [\tilde{v}''(x)]^\pm \geq 0,$$

it follows that w_\pm are convex in $[0, 1]$. In particular,

$$w_+(x) \leq (1-x)w_+(0) + xw_+(1), \quad x \in [0, 1].$$

We claim that the function

$$w(x) = \tilde{v}(1) - (1-x)\tilde{v}'(1) + (1-x)w_+(0) + xw_+(1) - w_-(x)$$

satisfies all the properties stated in the lemma. Indeed, it is clear that w is continuous in $[0, 1]$, \mathcal{C}^2 in $(0, 1]$, and $w(0) = u(0)$ since $w(0) = \tilde{v}(1) - \tilde{v}'(0) + w_+(0) - w_-(0) = \tilde{v}(0) = u(0)$. From the last inequality,

$$w(x) \geq \tilde{v}(1) - (1-x)\tilde{v}'(1) + w_+(x) - w_-(x) = \tilde{v}(x) \geq \tilde{u}(x) \geq u(x).$$

Finally, since

$$w''(x) = -(w_-(x))'' \leq 0$$

we have that w is concave, as required. \square

5. PROOF OF THE A PRIORI ESTIMATES

5.1. L^p estimates for the gradient. In this section we prove higher integrability properties of ∇w (Theorem 5.3) by using the extra hypotheses in Condition 2.10.

Lemma 5.1. *Under the hypotheses of Theorem 2.9, for all integers $\beta \geq 1$, every smooth solution w of (6) in Ω satisfies*

$$\begin{aligned} & \sum_{j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 |w_j|^{2\beta} \\ & \leq CC_1 \Lambda \left((A^4 + A^2 B^2) M_0^2 + A^2 \|f\|_{L^\infty(\tilde{\Gamma})} \right) \sum_{j=1}^n \int_{2\mathcal{R}} \xi^2 w_j^{2\beta-2} \\ & \quad + \mathcal{C}(C_1, \Lambda, M_0) \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2. \end{aligned}$$

Here C_1 as in Lemma 4.6, \mathcal{R} is any box such that $3\mathcal{R} \subset \Omega$ and \mathcal{R} satisfies property (ii) in the nondegeneracy Condition 2.3, $M_0 = \|w\|_{L^\infty(2\mathcal{R})}$, $\tilde{\Gamma} = 2\mathcal{R} \times [-M_0, M_0]$, and $B = B_\gamma(2\mathcal{R}, M_0)$ is as in Definition 2.7.

Proof. By the product rule,

$$\begin{aligned} |w_j|^{2\beta} &= \left(\partial_j (w w_j^{\beta-1}) - w \partial_j (w_j^{\beta-1}) \right)^2 \\ &\leq 2 \left(\partial_j (w w_j^{\beta-1}) \right)^2 + 2w^2 \left(\partial_j (w_j^{\beta-1}) \right)^2. \end{aligned}$$

Then, using that w is bounded by M_0 and applying Lemma 4.6 gives

$$\begin{aligned} & \sum_{j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 |w_j|^{2\beta} \\ & \leq 2 \sum_{j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 \left| \partial_j (w w_j^{\beta-1}) \right|^2 + 2M_0^2 \sum_{j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 \left| \partial_j (w_j^{\beta-1}) \right|^2 \\ & \leq CC_1 \Lambda \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} (w w_j^{\beta-1}) \right|^2 \\ & \quad + CC_1 \Lambda M_0^2 \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2 \\ (45) \quad & \leq CC_1 \Lambda \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_j^{2\beta-2} \\ & \quad + \mathcal{C}(C_1, \Lambda, M_0) \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2. \end{aligned}$$

Integrating by parts in the first term on the right, and using the fact that w is a solution of (6), we obtain

$$\begin{aligned} & \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_j^{2\beta-2} = - \sum_{j=1}^n \int_{2\mathcal{R}} w \operatorname{div} \left[|\nabla \zeta|^2 w_j^{2\beta-2} \mathcal{A}(x, w) \nabla w \right] \\ & = - \sum_{j=1}^n \int_{2\mathcal{R}} w w_j^{2\beta-2} \left(\nabla |\nabla \zeta|^2 \right) \cdot \mathcal{A}(x, w) \nabla w \\ & \quad - \sum_{j=1}^n \int_{2\mathcal{R}} w |\nabla \zeta|^2 \left(\nabla (w_j^{2\beta-2}) \right) \cdot \mathcal{A}(x, w) \nabla w \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \int_{2\mathcal{R}} w |\nabla \zeta|^2 w_j^{2\beta-2} (\vec{\gamma}(x, w) \cdot \nabla w + f(x, w)) \\
(46) \quad & = I + II + III.
\end{aligned}$$

In I , using the estimate

$$\left| (\nabla |\nabla \zeta|^2) \cdot \mathcal{A}(x, w) \nabla w \right| \leq \left| \nabla_{\sqrt{\mathcal{A}}, w} (|\nabla \zeta|^2) \right| \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right| \leq C A^2 |\nabla \zeta| \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|,$$

we have

$$\begin{aligned}
|I| & \leq M_0 \sum_{j=1}^n \int_{2\mathcal{R}} |w_j|^{2\beta-2} \left| (\nabla |\nabla \zeta|^2) \cdot \mathcal{A}(x, w) \nabla w \right| \\
(47) \quad & \leq \frac{1}{4} \sum_{j=1}^n \int_{2\mathcal{R}} |w_j|^{2\beta-2} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 + C A^4 M_0^2 \sum_{j=1}^n \int_{2\mathcal{R}} \xi^2 |w_j|^{2\beta-2}.
\end{aligned}$$

Using the identity $\nabla w_j^{2\beta-2} = 2w_j^{\beta-1} \nabla w_j^{\beta-1}$ in II , we obtain

$$\begin{aligned}
|II| & \leq 2M_0 \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 |w_j|^{\beta-1} \left| (\nabla w_j^{\beta-1}) \cdot \mathcal{A}(x, w) \nabla w \right| \\
(48) \quad & \leq \frac{1}{4} \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_j^{2\beta-2} + C M_0^2 \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2.
\end{aligned}$$

Finally, since $\vec{\gamma}$ is subunit with respect to \mathcal{A} in $\Gamma = \Omega \times \mathbb{R}$, we have $|\vec{\gamma}(x, w) \cdot \nabla w| \leq B \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|$ for all $x \in 2\mathcal{R}$, where $B = B(2\mathcal{R}, M_0)$. Then

$$\begin{aligned}
|III| & \leq M_0 \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 w_j^{2\beta-2} (|\vec{\gamma}(x, w) \cdot \nabla w| + |f(x, w)|) \\
(49) \quad & \leq \frac{1}{4} \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_j^{2\beta-2} \\
& \quad + \left(C B^2 M_0^2 + \|f\|_{L^\infty(\bar{\Gamma})} \right) \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 w_j^{2\beta-2}.
\end{aligned}$$

Combining (46), (47), (48) and (49), and absorbing into the left yields

$$\begin{aligned}
& \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_j^{2\beta-2} \\
& \leq C M_0^2 \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2 \\
& \quad + C \left((A^4 + A^2 B^2) M_0^2 + A^2 \|f\|_{L^\infty(\bar{\Gamma})} \right) \sum_{j=1}^n \int_{2\mathcal{R}} \xi^2 w_j^{2\beta-2}.
\end{aligned}$$

Using this estimate in the first term on the right of (45) finishes the proof of Lemma 5.1. \square

Lemma 5.2. *Under the hypothesis of Theorem 2.9, if w is a smooth solution of (6) in Ω , then for any $\beta \in \mathbb{N}$ and any box $\mathcal{R} \subset \Omega$ satisfying property (ii) in the nondegeneracy Condition 2.3,*

$$\begin{aligned}
& \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |w_j|^{2\beta} |\nabla_{\mathcal{A}, w} w|^2 \\
& \leq C B^2 \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\mathcal{A}, w} w_j^\beta \right|^2 + C B^2 \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 |\nabla w|^{2\beta}.
\end{aligned}$$

Here $B = B_\gamma(2\mathcal{R}, M_0)$ is as in Definition 2.7 and C depends on M_0 and $\|f\|_{L^\infty(\tilde{\Gamma})}$.

Proof. Integrating by parts, we have

$$\begin{aligned}
& \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta} |\nabla_{\mathcal{A},w} w|^2 \\
&= - \int_{2\mathcal{R}} w \operatorname{div} \left[\zeta^2 w_j^{2\beta} \mathcal{A}(x, w) \nabla w \right] \\
&= -2 \int_{2\mathcal{R}} w \zeta w_j^{2\beta} (\nabla \zeta) \cdot \mathcal{A}(x, w) \nabla w \\
&\quad -2 \int_{2\mathcal{R}} w \zeta^2 w_j^\beta (\nabla w_j^\beta) \cdot \mathcal{A}(x, w) \nabla w \\
&\quad + \int_{2\mathcal{R}} w \zeta^2 w_j^{2\beta} (\vec{\gamma}(x, w) \cdot \nabla w + f(x, w)) \\
(50) \quad &= I + II + III.
\end{aligned}$$

By Schwarz's inequality and since $\vec{\gamma}$ is of subunit type with respect to \mathcal{A} ,

$$\begin{aligned}
|I| &\leq CM_0^2 \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 w_j^{2\beta} + \frac{1}{6} \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w|^2 w_j^{2\beta} \\
|II| &\leq CM_0^2 \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_j^\beta|^2 + \frac{1}{6} \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w|^2 w_j^{2\beta} \\
|III| &\leq CB^2 M_0^2 \left(1 + \|f\|_{L^\infty(\tilde{\Gamma})}^2\right) \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta} + \frac{1}{6} \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w|^2 w_j^{2\beta}.
\end{aligned}$$

Applying these estimates to (50) and absorbing into the left gives

$$\begin{aligned}
& \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta} |\nabla_{\sqrt{\mathcal{A}},w} w|^2 \\
&\leq C \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\mathcal{A},w} w_j^\beta|^2 + C \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 w_j^{2\beta} + CB^2 \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta}
\end{aligned}$$

with C depending on M_0 and $\|f\|_{L^\infty(\tilde{\Gamma})}$. To obtain the conclusion of the lemma, we apply the Sobolev inequality (42) to the last term on the right and note that

$$CB^2 \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}},w} (\zeta w_j^\beta)|^2$$

is bounded by the sum of the first two terms on the right. \square

Theorem 5.3. *Under the hypothesis of Theorem 2.13, if w is a smooth solution of (6) in Ω , then for all integers $\beta \geq 1$ and every open Ω' with $\Omega' \Subset \Omega$, there exists a positive constant*

$$\mathcal{C}_\beta = \mathcal{C}_\beta \left(M_0, n, B, \Lambda, \vec{k}, \|f\|_{C^1(\Gamma')}, \|\vec{\gamma}\|_{C^1(\Gamma')}, \Omega, \operatorname{dist}(\Omega', \partial\Omega) \right)$$

such that

$$\sum_{j=1}^n \int_{\Omega'} w_j^{2\beta} + \sum_{j=1}^n \int_{\Omega'} |\nabla_{\sqrt{\mathcal{A}},w} w_j^\beta|^2 \leq \mathcal{C}_\beta.$$

Here $M_0 = \|w\|_{L^\infty(\Omega')}$, B denotes the constants $B_{\mathcal{A}}(\Omega', M_0)$, $B_\gamma(\Omega', M_0)$ in (16) and (18), and $\Gamma' = \Omega' \times [-M_0, M_0]$.

Proof. Let \mathcal{R} be a box satisfying property (ii) in the nondegeneracy Condition 2.3 and such that $2\mathcal{R} \Subset \Omega$. From Lemma 4.5,

$$\begin{aligned}
& \sum_{j=1}^n \int_{2\mathcal{R}} |\zeta \nabla_{\sqrt{\mathcal{A}},w} w_j^\beta|^2 \\
(51) \quad &\leq C\beta \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\mathcal{L}_w w_j) (w_j^{2\beta-1}) \right| + C \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 w_j^{2\beta}.
\end{aligned}$$

Differentiating (6) with respect to the j^{th} variable, we obtain

$$(52) \quad -\mathcal{L}_w w_j = \operatorname{div} \{ \partial_j \mathbf{A}(x) \} \nabla w + (\partial_j \vec{\gamma}) \cdot \nabla w + \vec{\gamma} \cdot \nabla w_j + \partial_j f$$

for all j , where \mathcal{L}_w is the linear operator (43), $\vec{\gamma} = \vec{\gamma}(x, w)$ and $\partial_j f = \partial[f(x, w)]$. Hence

$$\begin{aligned} & \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\mathcal{L}_w w_j) (w_j^{2\beta-1}) \right| \\ &= \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 \left(\operatorname{div} \{ \partial_j \mathbf{A}(x) \} \nabla w + (\partial_j \vec{\gamma}) \cdot \nabla w + \vec{\gamma} \cdot \nabla w_j + (\partial_j f) \right) (w_j^{2\beta-1}) \right| \\ &\leq \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 \left(\operatorname{div} \{ \partial_j \mathbf{A}(x) \} \nabla w \right) (w_j^{2\beta-1}) \right| \\ &\quad + \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 \left((\partial_j \vec{\gamma}) \cdot \nabla w + (\partial_j f) \right) (w_j^{2\beta-1}) \right| + \sum_{j=1}^n \left| \sum_{i=1}^n \int_{2\mathcal{R}} \zeta^2 \gamma^i w_{ij} w_j^{2\beta-1} \right| \\ (53) \quad &= I + II + III. \end{aligned}$$

From (12), (16) and the inequality

$$\sqrt{k^*(x, w)} |\partial_j u| \leq \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right|, \quad 1 \leq j \leq n,$$

where u is any smooth function, we get (with $B_{\mathcal{A}}$ as in (16))

$$\begin{aligned} |\{ \partial_j \mathbf{A}(x) \} \nabla u| &= |\{ \mathcal{A}_j + w_j \mathcal{A}_z \} \nabla u| \\ &\leq CB_{\mathcal{A}} \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right| + CB_{\mathcal{A}} \sqrt{k^*(x, w)} |w_j| \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right| \\ (54) \quad &\leq CB_{\mathcal{A}} \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right| + CB_{\mathcal{A}} \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right| \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right|. \end{aligned}$$

Writing

$$\nabla \left(\zeta^2 w_j^{2\beta-1} \right) = \frac{2\beta-1}{\beta} \zeta^2 w_j^{\beta-1} \nabla w_j^\beta + w_j^{2\beta-1} \nabla \zeta^2,$$

integrating by parts, and using $\frac{2\beta-1}{\beta} \leq 2$ and (54), we obtain

$$\begin{aligned} I &= \sum_{j=1}^n \left| \int_{2\mathcal{R}} (\nabla w) \cdot \{ \partial_j \mathbf{A}(x) \} \nabla \left(\zeta^2 w_j^{2\beta-1} \right) \right| \\ &\leq C \sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_j^{\beta-1} (\nabla w) \cdot \{ \partial_j \mathbf{A}(x) \} \nabla w_j^\beta \right| \\ &\quad + \sum_{j=1}^n \left| \int_{2\mathcal{R}} w_j^{2\beta-1} (\nabla w) \cdot \{ \partial_j \mathbf{A}(x) \} \nabla \zeta^2 \right| \\ &\leq CB_{\mathcal{A}} \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |w_j|^{\beta-1} |\nabla w| \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta \right| \left\{ 1 + \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right| \right\} \\ &\quad + CB_{\mathcal{A}} \sum_{j=1}^n \int_{2\mathcal{R}} \zeta |w_j|^{2\beta-1} |\nabla w| \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right| \left\{ 1 + \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right| \right\} \\ (55) \quad &\leq \alpha \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta \right|^2 + C \int_{2\mathcal{R}} |\nabla w|^{2\beta} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 \\ &\quad + \frac{CB_{\mathcal{A}}^2}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} + \frac{CB_{\mathcal{A}}^2}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2. \end{aligned}$$

From the chain rule, the super subordination Condition (18) for γ , and Young's inequality,

$$\begin{aligned}
II &\leq \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left(|f_j| + |w_j| |f_z| + \sum_{i=1}^n |\gamma_j^i| |w_i| \right) |w_j|^{2\beta-1} \\
&\quad + \sum_{j=1}^n \left| \int_{2\mathcal{R}} \sum_{i=1}^n \zeta^2 w_i (\gamma_z^i) w_j^{2\beta} \right| \\
&\leq C_a \int_{2\mathcal{R}} \zeta^2 (|\nabla w|^{2\beta-1} + |\nabla w|^{2\beta}) + C \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} w|^2 \\
(56) \quad &\leq C_a \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} + C \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} w|^2 + C_a,
\end{aligned}$$

where $C_a = C_a(B_\gamma, \Lambda, \|f\|_{C^1(\bar{\Gamma})}, \|\tilde{\gamma}\|_{C^1(\bar{\Gamma})})$ with B_γ as in (18).

Since $\tilde{\gamma}$ is of subunit type with respect to \mathcal{A} ,

$$\begin{aligned}
III &= \frac{1}{\beta} \sum_{j=1}^n \left| \sum_{i=1}^n \int_{2\mathcal{R}} (\zeta \gamma^i \partial_i (w_j^\beta)) (\zeta w_j^\beta) \right| \\
(57) \quad &\leq \alpha \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta|^2 + \frac{B_\gamma^2}{\alpha} \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta}.
\end{aligned}$$

Using (55), (56) and (57) in (53) yields

$$\begin{aligned}
&\sum_{j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\mathcal{L}_w w_j) w_j^{2\beta-1} \right| \\
&\leq C\alpha \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta|^2 + C \int_{2\mathcal{R}} |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} \zeta|^2 \\
&\quad + \frac{C_a B^2}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} + \frac{C B_{\mathcal{A}}^2}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} w|^2 + C_a.
\end{aligned}$$

Substituting on the right of (51) and absorbing into the left, we get

$$\begin{aligned}
(58) \quad \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta|^2 &\leq C(\beta+1) \int_{2\mathcal{R}} |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} \zeta|^2 \\
&\quad + C_a \beta B^2 \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} \\
&\quad + C \beta B_{\mathcal{A}}^2 \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} w|^2 + \beta C_a.
\end{aligned}$$

Now we will further assume that for fixed constants C and β , the constant $B_{\mathcal{A}}$ is small enough so that

$$(59) \quad C \beta B_{\mathcal{A}}^2 B^2 \leq \frac{1}{2}.$$

We will show later that this assumption causes no loss of generality. Applying Lemma 5.2 to the third term on the right of (58), and absorbing into the left using (59), we obtain

$$\begin{aligned}
\sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta|^2 &\leq C(\beta+1) \int_{2\mathcal{R}} |\nabla w|^{2\beta} |\nabla_{\sqrt{\mathcal{A}}, w} \zeta|^2 + C_a \beta B^2 \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{2\beta} \\
&\quad + C \beta B_{\mathcal{A}}^2 B^2 \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}}, w} \zeta|^2 |\nabla w|^{2\beta} + \beta C_a
\end{aligned}$$

where C depends on M_0 and $\|f\|_{L^\infty(\bar{\Gamma})}$. In turn, applying the Sobolev inequality (42) to each ζw_j^β in the second term on the right, and applying Lemma 5.1 to the first and third terms on the right

(and the similar term arising from the Sobolev inequality), we obtain

$$(60) \quad \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta \right|^2 \leq \beta \mathcal{C}_a + \mathcal{C}_b \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{\beta-1} \right|^2 + \mathcal{C}_b \sum_{j=1}^n \int_{2\mathcal{R}} \xi^2 w_j^{2\beta-2},$$

after absorbing into the left, by taking r^1 small enough depending on β , B , Λ , $\|f\|_{C^1(\bar{\Gamma})}$ and $\|\tilde{\gamma}\|_{C^1(\bar{\Gamma})}$; here it is important to note that the constants multiplying r^1 do not depend on the size of \mathcal{R} . Also,

$$\mathcal{C}_b = \mathcal{C} \left(M_0, \vec{k}, \mathcal{R}, \beta, n, B, \Lambda, \|f\|_{C^1(\bar{\Gamma})}, \|\tilde{\gamma}\|_{C^1(\bar{\Gamma})} \right).$$

The conclusion of the theorem will now follow by induction from (60) and the Sobolev inequality. Indeed, if $\beta = 1$,

$$\sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j \right|^2 \leq \beta \mathcal{C}_a + \mathcal{C}_b \int_{2\mathcal{R}} \xi^2 \leq \mathcal{C}_b.$$

By the Sobolev inequality (42) and Lemma 5.1,

$$(61) \quad \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^2 + \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j \right|^2 \leq \mathcal{C}_b.$$

Since $\Omega' \Subset \Omega$, there are a finite number of rectangles $\{\mathcal{R}_i\}$ satisfying (61) and such that $\Omega' \subset \bigcup \mathcal{R}_i$. Choosing \mathcal{C}_1 as in Theorem 5.3 in case $\beta = 1$, it follows that

$$\sum_{j=1}^n \int_{\Omega'} w_j^2 + \sum_{j=1}^n \int_{\Omega'} \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j \right|^2 \leq \mathcal{C}_1.$$

Suppose now that we have shown that

$$(62) \quad \sum_{j=1}^n \int_{\Omega'} w_j^{2\beta} + \sum_{j=1}^n \int_{\Omega'} \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^\beta \right|^2 \leq \mathcal{C}_\beta.$$

for $\beta = 1, 2, \dots, M$. Given $x \in \Omega'$, let \mathcal{R} be a box satisfying property (ii) in Condition 2.3 and such that $x \in \frac{1}{3}\mathcal{R} \subset 2\mathcal{R} \subset \Omega' \Subset \Omega$. Then, from (60),

$$\begin{aligned} \sum_{j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^{M+1} \right|^2 &\leq (M+1)\mathcal{C}_a + \mathcal{C}_b \sum_{j=1}^n \int_{2\mathcal{R}} |\nabla \zeta|^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^M \right|^2 \\ &\quad + \mathcal{C}_b \int_{2\mathcal{R}} \xi^2 |\nabla w|^{2M} \\ &\leq (M+1)\mathcal{C}_a + A^2 \mathcal{C}_b \sum_{j=1}^n \int_{\Omega'} \left| \nabla_{\sqrt{\mathcal{A}}, w} w_j^M \right|^2 + \mathcal{C}_b \int_{\Omega'} |\nabla w|^{2M} \leq \mathcal{C}_{M+1}. \end{aligned}$$

An application of the Sobolev inequality, Lemma 5.1, and a covering argument complete the induction.

It remains to show that the extra assumption (59) imposes no loss of generality. For a constant $M \geq 1$, let $u(x) = Mw(x)$ and

$$\begin{aligned} \tilde{\mathcal{A}}(x, q) &= \mathcal{A}(x, q/M), & \tilde{k}^*(x, q) &= k^*(x, q/M), \\ \tilde{\gamma}(x, q) &= \gamma(x, q/M), & \tilde{f}(x, q) &= Mf(x, q/M). \end{aligned}$$

Then u satisfies

$$\begin{aligned} &\operatorname{div} \tilde{\mathcal{A}}(x, u) \nabla u + \tilde{\gamma}(x, u) \cdot \nabla u + \tilde{f}(x, u) \\ &= \operatorname{div} \tilde{\mathcal{A}}(x, Mw(x)) \nabla Mw(x) + \tilde{\gamma}(x, Mw(x)) \cdot \nabla Mw(x) + \tilde{f}(x, Mw(x)) \\ &= M [\operatorname{div} \mathcal{A}(x, w(x)) \nabla w(x) + \gamma(x, w(x)) \cdot \nabla w(x) + f(x, w(x))] = 0. \end{aligned}$$

On the other hand, by (16),

$$\begin{aligned}
\left| \partial_q \tilde{\mathcal{A}}(x, q) \xi \right|^2 &= \left| \partial_q \mathcal{A}(x, q/M) \xi \right|^2 = M^{-2} \left| \mathcal{A}_z(x, q/M) \xi \right|^2 \\
&\leq M^{-2} B_{\mathcal{A}}^2 k^*(x, q/M) \xi^t \mathcal{A}(x, q/M) \xi \\
&= B_{\mathcal{A}^2} M^{-2} \tilde{k}^*(x, q) \xi^t \tilde{\mathcal{A}}(x, q) \xi \\
&= \left(\tilde{B}_{\tilde{\mathcal{A}}} \right)^2 \tilde{k}^*(x, q) \xi^t \tilde{\mathcal{A}}(x, q) \xi
\end{aligned}$$

with $\tilde{B}_{\tilde{\mathcal{A}}} = B_{\mathcal{A}} M^{-1}$. Since γ is of subunit type (Definition 2.7),

$$\left(\sum_{i=1}^n \tilde{\gamma}^i(x, q) \xi_i \right)^2 = \left(\sum_{i=1}^n \gamma^i(x, q/M) \xi_i \right)^2 \leq B_{\gamma}^2 \xi^t \mathcal{A}(x, q/M) \xi = B_{\gamma}^2 \xi^t \tilde{\mathcal{A}}(x, q) \xi.$$

Also, by (18),

$$\begin{aligned}
\left| \partial_q \tilde{\gamma}(x, q) \cdot \xi \right|^2 &= M^{-2} \left| \tilde{\gamma}_z(x, q/M) \cdot \xi \right|^2 \leq M^{-2} B_{\gamma}^2 k^*(x, q/M) \xi^t \mathcal{A}(x, q/M) \xi \\
&= (B_{\gamma} M^{-1})^2 \tilde{k}^*(x, q) \xi^t \tilde{\mathcal{A}}(x, q) \xi.
\end{aligned}$$

Hence $u(x)$ satisfies the equation

$$\operatorname{div} \tilde{\mathcal{A}}(x, u) \nabla u + \tilde{\gamma}(x, u) \cdot \nabla u + \tilde{f}(x, u) = 0$$

with the corresponding constants in Condition (2.10), namely

$$\tilde{B}_{\tilde{\mathcal{A}}} = B_{\mathcal{A}} M^{-1} \quad \text{and} \quad \tilde{B}_{\gamma} = B_{\gamma}.$$

Hence, taking M large enough and letting $\tilde{B} = \max\{B_{\mathcal{A}} M^{-1}, B_{\gamma}\}$, we have that

$$C_{\beta} \left(\tilde{B}_{\tilde{\mathcal{A}}} \right)^2 \tilde{B}^2 = C_{\beta} \left(\frac{B_{\mathcal{A}}}{M} \right)^2 \max \left\{ \left(\frac{B_{\mathcal{A}}}{M} \right)^2, B_{\gamma}^2 \right\} \leq \frac{1}{2},$$

so the extra assumption (59) holds for the operator $\operatorname{div} \tilde{\mathcal{A}}(x, \cdot) \nabla + \tilde{\gamma}(x, \cdot) \cdot \nabla + \tilde{f}(x, \cdot)$. By the previous calculations, there is a constant

$$\tilde{\mathcal{C}}_{\beta} = \tilde{\mathcal{C}}_{\beta} \left(\tilde{M}_0, n, \tilde{B}, \Lambda, \tilde{k}, \left\| \tilde{f} \right\|_{C^1(\tilde{\Gamma}')} , \left\| \tilde{\gamma} \right\|_{C^1(\tilde{\Gamma}')} , \Omega, \operatorname{dist}(\Omega', \partial\Omega) \right)$$

such that

$$\sum_{j=1}^n \int_{\Omega'} u_j^{2\beta} + \sum_{j=1}^n \int_{\Omega'} \left| \nabla \sqrt{\tilde{\mathcal{A}}_{\tilde{u}}} u_j^{\beta} \right|^2 \leq \tilde{\mathcal{C}}_{\beta};$$

here $\tilde{M}_0 = \|u\|_{L^{\infty}(\Omega')} = M \|w\|_{L^{\infty}(\Omega')}$ and $\tilde{\Gamma}' = \Omega' \times [-\tilde{M}_0, \tilde{M}_0]$. The general result for w follows from the identity $w = Mu$ and the definitions of \tilde{f} and $\tilde{\gamma}$. \square

5.2. Proof of Theorem 2.13. In this section we prove the a priori estimate in Theorem 2.13 as a consequence of the higher integrability of ∇w established in the previous section. Theorem 2.13 will be the main tool in the proof of Theorem 2.18.

By Theorem 2.9, we only need to show that ∇w is locally bounded in terms of appropriate parameters, i.e., we only need to show that for every box $\mathcal{R} \subset 4\mathcal{R} \subset \Omega' \Subset \Omega$,

$$\|\nabla w\|_{L^{\infty}(\mathcal{R})} \leq \mathcal{C} \left(\|w\|_{L^{\infty}(\Omega')}, \mathcal{R}, \vec{k}, \mathcal{A}, f, \vec{\gamma} \right);$$

the dependence of \mathcal{C} on its arguments will be made more explicit as we proceed. By the Sobolev imbedding theorem, it is enough to show that for some $\beta > n$,

$$(63) \quad \|w_{ij}\|_{L^{\beta}(2\mathcal{R})} \leq \mathcal{C}_{\beta} \left(\|w\|_{L^{\infty}(\Omega')}, \mathcal{R}, \vec{k}, \mathcal{A}, f, \vec{\gamma} \right), \quad 1 \leq i, j \leq n.$$

Let $\mathcal{R} \subset 4\mathcal{R} \subset \Omega'$ be a box satisfying property (ii) in the nondegeneracy Condition 2.3. Since such boxes cover Ω' , there is no loss of generality in adding this extra condition. Applying the

Caccioppoli inequality for the \mathcal{A} -gradient, Lemma 4.5, to the smooth functions $w_{ij} = \partial_i \partial_j w$, we get for $\beta \in \mathbb{N}$ that

$$\begin{aligned} & \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|^2 \\ & \leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\mathcal{L}_w w_{ij}) w_{ij}^{2\beta-1} \right| + C \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^{2\beta}. \end{aligned}$$

We estimate the first term on the right by Lemma 4.8, obtaining

$$\begin{aligned} & \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|^2 \\ & \leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} (\nabla w_j) \cdot \mathbf{A}_i \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i (\nabla w_j) \cdot \mathbf{A}_z \nabla \zeta^2 w_{ij}^{2\beta-1} \right| \\ & \quad + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_{ij} (\nabla w) \cdot \mathbf{A}_z \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + \frac{C_0\beta}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla^2 w|^{2\beta} \\ & \quad + C\beta\alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \vec{\gamma} \cdot \nabla w_{ij}^\beta \right|^2 + C\beta\alpha \int_{2\mathcal{R}} \zeta^2 |\vec{\gamma}_z \cdot \nabla w|^2 |\nabla^2 w|^{2\beta} \\ & \quad + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 ((\partial_j \vec{\gamma}) \cdot \nabla w_i) w_{ij}^{2\beta-1} \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i (\nabla w) \cdot \mathbf{A}_{jz} \nabla \zeta^2 w_{ij}^{2\beta-1} \right| \\ & \quad + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_i w_j (\nabla w) \cdot \mathbf{A}_{zz} \nabla \zeta^2 w_{ij}^{2\beta-1} \right| + C_0\beta \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{6\beta} \\ & \quad + C \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^{2\beta} + C_0\beta \\ (64) \quad & = I + II + \dots + VIII + IX + C_0\beta \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{6\beta} \\ & \quad + C \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^{2\beta} + C_0\beta, \end{aligned}$$

where $0 < \alpha < 1$ is arbitrary, $\vec{\gamma} = \vec{\gamma}(x, w)$, $f = f(x, w)$, $\mathbf{A} = \mathcal{A}(x, w)$, $\mathbf{A}_i = \mathcal{A}_i(x, w)$, etc., and

$$C_0 = C_0 \left(\|\mathcal{A}\|_{C^3(\Gamma')}, \|f\|_{C^2(\Gamma')}, \|\vec{\gamma}\|_{C^2(\Gamma')} \right)$$

with $\Gamma' = \Omega' \times [-M_0, M_0]$ and $M_0 = \|w\|_{L^\infty(\Omega')}$.

Note that

$$\partial_i \mathbf{A}(x) = \partial_i \mathcal{A}(x, w) = \mathcal{A}_i(x, w) + w_i \mathcal{A}_z(x, w) = \mathbf{A}_i + w_i \mathbf{A}_z.$$

If u is any smooth function, from (12), (16), (17), (18) and since $\vec{\gamma}$ is of subunit type with respect to \mathcal{A} , it follows that

$$(65) \quad |\mathbf{A}_i \nabla u|^2 \leq B_{\mathcal{A}}^2 \left| \nabla_{\sqrt{\mathcal{A}},w} u \right|^2,$$

$$\begin{aligned} |w_i \mathbf{A}_z \nabla u|^2 & \leq B_{\mathcal{A}}^2 |w_i|^2 k^*(x, w) \sum_{j=1}^n k^j(x, w) (\partial_j u)^2 \\ (66) \quad & \leq B_{\mathcal{A}}^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w \right|^2 \left| \nabla_{\sqrt{\mathcal{A}},w} u \right|^2, \end{aligned}$$

$$(67) \quad \sum_{i=1}^n |\mathbf{A}_{iz} \nabla u|^2 + |\mathbf{A}_{zz} \nabla u|^2 \leq (B'_{\mathcal{A}})^2 \left| \nabla_{\sqrt{\mathcal{A}},w} u \right|^2,$$

$$(68) \quad |\vec{\gamma} \cdot \nabla u|^2 \leq B_\gamma^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right|^2 \quad \text{and}$$

$$(69) \quad \sum_{i=1}^n |\vec{\gamma}_i \cdot \nabla u|^2 + |\vec{\gamma}_z \cdot \nabla u|^2 \leq B_\gamma^2 k^*(x, w) \left| \nabla_{\sqrt{\mathcal{A}}, w} u \right|^2.$$

We will now apply these estimates together with the Schwarz and triangle inequalities to treat each term of (64). We will incorporate the constants $B_{\mathcal{A}}, B_\gamma$, etc. in our generic constant C .

By definition of I and (65),

$$\begin{aligned} I &\leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta w_{ij}^{2\beta-1} (\nabla w_j) \cdot \mathbf{A}_i (\nabla \zeta) \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_{ij}^{\beta-1} (\nabla w_j) \cdot \mathbf{A}_i \nabla w_{ij}^\beta \right| \\ &\leq \alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta} + \frac{C\beta^2}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta}. \end{aligned}$$

Similarly, using (66),

$$\begin{aligned} II &\leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta w_i w_{ij}^{2\beta-1} (\nabla w_j) \cdot \mathbf{A}_z (\nabla \zeta) \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_{ij}^{\beta-1} w_i (\nabla w_j) \cdot \mathbf{A}_z \nabla w_{ij}^\beta \right| \\ &\leq \alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta} \\ &\quad + \frac{C\beta^2}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_{ij}^{2\beta}. \end{aligned}$$

Treating III analogously, we get

$$\begin{aligned} (70) \quad &I + II + III + IV \\ &\leq \alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 |w_{ij}|^{2\beta} \\ &\quad + \frac{C\beta^2}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + \frac{C\beta^2}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_{ij}^{2\beta}. \end{aligned}$$

By (68) and (69),

$$(71) \quad V + VI \leq C\beta\alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + C\beta\alpha \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_{ij}^{2\beta}.$$

Now, using the identity $\partial_j \vec{\gamma} = \vec{\gamma}_j + w_j \vec{\gamma}_z$,

$$\begin{aligned} VII &\leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\vec{\gamma}_j \cdot \nabla w_i) w_{ij}^{2\beta-1} \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_j (\vec{\gamma}_z \cdot \nabla w_i) w_{ij}^{2\beta-1} \right| \\ &= VII_1 + VII_2. \end{aligned}$$

We have

$$\begin{aligned} VII_1 &\leq C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \|\vec{\gamma}\|_{C^1(2\mathcal{R})} w_{ij}^{2\beta} \\ &\leq C_0\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta}. \end{aligned}$$

Now, integrating by parts,

$$VII_2 = C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \nabla w \cdot \partial_i \left(\zeta^2 \vec{\gamma}_z w_j w_{ij}^{2\beta-1} \right) \right|$$

$$\begin{aligned}
&\leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta w_j \zeta_i w_{ij}^{2\beta-1} (\nabla w) \cdot \vec{\gamma}_z \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_j w_{ij}^{2\beta-1} (\nabla w) \cdot (\partial_i \vec{\gamma}_z) \right| \\
&\quad + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\nabla w) \cdot \vec{\gamma}_z w_{ij}^{2\beta} \right| + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_j w_{ij}^{\beta-1} (\nabla w) \cdot \vec{\gamma}_z \left(\partial_i w_{ij}^\beta \right) \right| \\
&= VII_{2,1} + VII_{2,2} + VII_{2,3} + VII_{2,4}.
\end{aligned}$$

Then

$$\begin{aligned}
VII_{2,1} &= C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta w_j w_{ij}^{2\beta-1} \zeta_i \vec{\gamma}_z \cdot (\nabla w) \right| \\
(72) \quad &\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_j^2 \zeta_i^2 w_{ij}^{2\beta-2} |\vec{\gamma}_z \cdot \nabla w|^2.
\end{aligned}$$

Assume for the moment that $\beta > 1$. Then by using the super subordination Condition 2.10 for γ , and Young's inequality in the form

$$\int |fg| \leq \frac{\beta-1}{\beta} \int |f|^{\frac{2\beta}{2\beta-2}} + \frac{1}{\beta} \int |g|^\beta,$$

the second term on the right of (72) is bounded by

$$\begin{aligned}
&C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_j^2 w_{ij}^{2\beta-2} \zeta_i^2 k^* \left| \nabla_{\sqrt{\mathcal{A}},w} w \right|^2 \\
&\leq C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta-2} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 |\nabla w|^4 \\
&\leq C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 + C\alpha\beta A^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \xi^2 |\nabla w|^{4\beta},
\end{aligned}$$

where we used that $\left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 \leq CA^2 \xi^2$. If on the other hand $\beta = 1$, the estimation of the second term in (72) is simpler; without using Young's inequality, we can estimate it by just the second term above. Thus,

$$\begin{aligned}
VII_{2,1} &\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 \\
&\quad + C\alpha\beta A^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \xi^2 |\nabla w|^{4\beta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
VII_{2,2} &\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^{2\beta-2} |\nabla w|^4 \\
&\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C_0\alpha\beta \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{4\beta},
\end{aligned}$$

and

$$VII_{2,3} \leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w \right|^2 w_{ij}^{2\beta}.$$

Now, from (69),

$$\begin{aligned}
\left| (\nabla w) \cdot \vec{\gamma}_z \left(\partial_i w_{ij}^\beta \right) \right| &\leq B_\gamma \left| \nabla_{\sqrt{\mathcal{A}},w} w \right| \sqrt{k^*(x,w)} \left| \partial_i w_{ij}^\beta \right| \\
&\leq B_\gamma \left| \nabla_{\sqrt{\mathcal{A}},w} w \right| \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|.
\end{aligned}$$

Thus,

$$\begin{aligned} VII_{2,4} &\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_j^2 w_{ij}^{2\beta-2} |\nabla_{\sqrt{\mathcal{A}},w} w|^2 + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2 \\ &\leq \frac{C\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + \frac{C_0\beta}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{4\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2. \end{aligned}$$

Assembling all these estimates, we obtain

$$\begin{aligned} (73) \quad VII &\leq \frac{C_0\beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 \\ &\quad + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w|^2 w_{ij}^{2\beta} \\ &\quad + \frac{C_0 A^2 \beta}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{4\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2. \end{aligned}$$

By (67),

$$\begin{aligned} (74) \quad VIII &\leq C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta w_i w_{ij}^{2\beta-1} (\nabla w) \cdot \mathbf{A}_{jz} (\nabla \zeta) \right| \\ &\quad + C\beta \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 w_i (\nabla w) w_{ij}^{\beta-1} \cdot \mathbf{A}_{jz} \nabla w_{ij}^\beta \right| \\ &\leq C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2 \\ &\quad + \frac{C(B'_A)^2 \beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_i^2 w_{ij}^{2\beta-2} |\nabla w|^2 \\ &\leq C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 + \frac{C(B'_A)^2 \beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \\ &\quad + \frac{C_0(B'_A)^2 \beta}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{4\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} (75) \quad IX &\leq C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} w_{ij}^{2\beta} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 + \frac{C_0(B'_A)^2 \beta}{\alpha} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \\ &\quad + \frac{C(B'_A)^2 \beta}{\alpha} \int_{2\mathcal{R}} \zeta^2 |\nabla w|^{6\beta} + C\alpha\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2. \end{aligned}$$

Using estimates (70), (71), (73), (74) and (75) in (64), and absorbing into the left, we obtain

$$\begin{aligned} (76) \quad \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta|^2 &\leq C\beta \sum_{i,j=1}^n \int_{2\mathcal{R}} |\nabla_{\sqrt{\mathcal{A}},w} \zeta|^2 w_{ij}^{2\beta} \\ &\quad + C_0\beta^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \\ &\quad + C\beta^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 |\nabla_{\sqrt{\mathcal{A}},w} w|^2 w_{ij}^{2\beta} \end{aligned}$$

$$+ \mathcal{C}_0 \beta^2 \int_{2\mathcal{R}} \xi^2 |\nabla w|^{6\beta} + \mathcal{C}_0 A^2 \beta.$$

We now estimate the third term on the right of (76) proceeding as in the proof of Lemma 5.2. Integrating by parts,

$$\begin{aligned} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_{ij}^{2\beta} &= - \sum_{i,j=1}^n \int_{2\mathcal{R}} w \operatorname{div} \left\{ \zeta^2 w_{ij}^{2\beta} \mathcal{A}(x, w) \nabla w \right\} \\ &\leq M_0 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} w_{ij}^{2\beta} (\nabla \zeta^2) \cdot \mathcal{A}(x, w) \nabla w \right| \\ &\quad + M_0 \sum_{i,j=1}^n \left| \int_{2\mathcal{R}} \zeta^2 (\nabla w_{ij}^{2\beta}) \cdot \mathcal{A}(x, w) \nabla w \right| \\ &\quad + M_0 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} |\vec{\gamma} \cdot \nabla w + f|. \end{aligned}$$

By Schwarz's inequality, the identity $\nabla w_{ij}^{2\beta} = 2w_{ij}^\beta \nabla w_{ij}^\beta$, and (68), we obtain after absorbing into the left,

$$\begin{aligned} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w \right|^2 w_{ij}^{2\beta} &\leq C(M_0^2 + 1) \left(\|f\|_{L^\infty(\Gamma')} + B_\gamma^2 \right) \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \\ &\quad + CM_0^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta} + CM_0^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2. \end{aligned}$$

Using this on the right of (76) gives

$$\begin{aligned} (77) \quad \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 &\leq C\beta^2 (M_0^2 + 1) \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta} \\ &\quad + \mathcal{C}_1 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \\ &\quad + C\beta^2 M_0^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 \\ &\quad + \mathcal{C}_0 B^2 \beta^2 \int_{2\mathcal{R}} \xi^2 |\nabla w|^{6\beta} + \mathcal{C}_0 A^2 \beta, \end{aligned}$$

with

$$\mathcal{C}_1 = \mathcal{C}_1 \left(M_0, B, \beta, \|\mathcal{A}\|_{C^3(\Gamma')}, \|f\|_{C^2(\Gamma')}, \|\vec{\gamma}\|_{C^2(\Gamma')} \right).$$

By the Sobolev inequality (42) and the product rule,

$$\int_{2\mathcal{R}} \zeta^2 w_{ij}^{2\beta} \leq C(r^1)^2 \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + C(r^1)^2 \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta}.$$

Applying this to the second term on the right of (77), and taking r^1 small enough depending on $M_0, \beta, B, \|\mathcal{A}\|_{C^3(\Gamma')}, \|f\|_{C^2(\Gamma')}$ and $\|\vec{\gamma}\|_{C^2(\Gamma')}$ (in order to absorb the term resulting from the first term on the right of the last estimate), we get

$$\begin{aligned} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 &\leq \mathcal{C}_1 \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}}, w} \zeta \right|^2 w_{ij}^{2\beta} \\ &\quad + C\beta^2 M_0^2 \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}}, w} w_{ij}^\beta \right|^2 + \mathcal{C}_0 B^2 \beta^2 \int_{2\mathcal{R}} \xi^2 |\nabla w|^{6\beta} + \mathcal{C}_0 A^2 \beta. \end{aligned}$$

Now restrict $1 \leq \beta \leq n+1$, and assume that M_0 is small enough so that

$$(78) \quad C\beta^2 M_0^2 \leq C(n+1)^2 M_0^2 \leq 1/2.$$

Then the second term on the right above may be absorbed into the left side to obtain

$$(79) \quad \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|^2 \leq C_1 \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^{2\beta} + C_0 B^2 \beta^2 \int_{2\mathcal{R}} \xi^2 |\nabla w|^{6\beta} + C_0 A^2 \beta.$$

For any $q \geq 1$,

$$\|\chi \nabla \mathbf{A}\|_{L^q} \leq C \left(\int_{2\mathcal{R}} \sum_{i=1}^n |\mathbf{A}_i|^q + |\nabla w|^q |\mathbf{A}_z|^q \right)^{\frac{1}{q}} \leq C \|\mathcal{A}\|_{C^1(\Gamma')} \left(1 + \|\nabla w\|_{L^q(2\mathcal{R})} \right).$$

Then from Lemma 4.7 applied to the function $u = \xi w_{ij}$, choosing $q = n+1$, there exists $1 < p = p(n) < 2$ such that

$$\begin{aligned} \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^{2\beta} &= \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 (\xi w_{ij})^{2\beta} \\ &\leq \varepsilon^{-1} \mathcal{C} \left(n, A, \|\nabla w\|_{L^{n+1}(2\mathcal{R})} \right) \sum_{i,j=1}^n \left\| \xi w_{ij}^\beta \right\|_{L^p}^2 + C\varepsilon \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|^2 \end{aligned}$$

for all $1 \leq \beta \leq n+1$. On the other hand, by Theorem 5.3,

$$(80) \quad \|\nabla w\|_{L^{n+1}(2\mathcal{R})} + \|\nabla w\|_{L^{6\beta}(2\mathcal{R})} \leq C_2$$

where $C_2 = C_2 \left(M_0, n, B, \Lambda, \vec{k}, \|f\|_{C^1(\Gamma')}, \|\vec{\gamma}\|_{C^1(\Gamma')}, \Omega, \text{dist}(\Omega', \partial\Omega) \right)$. Using these estimates in (79) and absorbing into the left gives

$$(81) \quad \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij}^\beta \right|^2 \leq C_3 \sum_{i,j=1}^n \left\| \xi w_{ij}^\beta \right\|_{L^p}^2 + C_3.$$

with

$$C_3 = C_3 \left(M_0, n, B, \Lambda, \vec{k}, \mathcal{R}, \|\mathcal{A}\|_{C^3(\Gamma')}, \|f\|_{C^2(\Gamma')}, \|\vec{\gamma}\|_{C^2(\Gamma')}, \Omega, \text{dist}(\Omega', \partial\Omega) \right),$$

where we have used Remark 4.4 to substitute the dependence on A by dependence on $\vec{k}, \mathcal{R}, M_0$.

Choosing $\beta = 1$ in (79) and applying Lemma 4.6 and (80) gives

$$\begin{aligned} \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij} \right|^2 &\leq C_1 \sum_{i,j=1}^n \int_{2\mathcal{R}} \left| \nabla_{\sqrt{\mathcal{A}},w} \zeta \right|^2 w_{ij}^2 + C_0 B^2 \int_{2\mathcal{R}} \xi^2 |\nabla w|^6 + C_0 A^2 \\ &\leq C_1 C_1 A^2 \Lambda \sum_{i=1}^n \int_{2\mathcal{R}} \xi^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_i \right|^2 + C_3. \end{aligned}$$

Estimating the first term on the right by Theorem 5.3, we then get

$$(82) \quad \sum_{i,j=1}^n \int_{2\mathcal{R}} \zeta^2 \left| \nabla_{\sqrt{\mathcal{A}},w} w_{ij} \right|^2 \leq C_3.$$

To finish the proof, we iterate (81) in a similar fashion as in the proof of Theorem 5.3, using (82) to start the iteration. We omit the details. As a result, we obtain

$$\sum_{i,j=1}^n \int_{\Omega'} |w_{ij}|^{n+1} \leq C^* \left(M_0, n, B, \Lambda, \vec{k}, \|\mathcal{A}\|_{C^3(\Gamma')}, \|f\|_{C^2(\Gamma')}, \|\vec{\gamma}\|_{C^2(\Gamma')}, \text{dist}(\Omega', \partial\Omega), \Omega \right).$$

From the Sobolev embedding theorem it follows that

$$\|\nabla w\|_{L^\infty(\Omega')} \leq C^*,$$

as desired.

It remains to prove that assumption (78) does not result in a loss of generality. This will be accomplished by a change of variables as at the end of the proof of Theorem 5.3. In fact, letting $u(x) = w(x)/N$, u satisfies the equation

$$(83) \quad \operatorname{div} \tilde{\mathcal{A}}(x, u) \nabla u + \tilde{\gamma}(x, u) \cdot \nabla u + \tilde{f}(x, u) = 0$$

in Ω , where

$$\tilde{\mathcal{A}}(x, q) = \mathcal{A}(x, Nq), \quad \tilde{\gamma}(x, q) = \gamma(x, Nq), \quad \tilde{f}(x, q) = \frac{1}{N} f(x, Nq).$$

Moreover,

$$\tilde{M}_0 = \|u\|_{L^\infty(\Omega')} = \frac{M_0}{N}.$$

Hence, for N big enough, the analogue of (78) holds for u . The result for w then follows from the identity $w = Nu$.

6. PROOF OF THE HYPOELLIPTICITY THEOREM

In this section we prove our main results Theorems 2.18 and 2.17. To do so, we will apply Theorem 15.19 of [6]. For easy reference we now state a special version of this theorem suitable for our needs.

Theorem 6.1 (Theorem 15.19 in [6]). *Let Ω be a bounded domain in \mathbb{R}^n satisfying an exterior sphere condition at each point of $\partial\Omega$. Let \mathcal{M} be a divergence structure operator,*

$$\mathcal{M}(w) = \operatorname{div} \mathcal{S}(x, w, \nabla w) + \mathcal{T}(x, w, \nabla w),$$

where $\mathcal{S} = (\mathcal{S}^1, \mathcal{S}^2, \dots, \mathcal{S}^n)$, $\mathcal{S}^i \in \mathcal{C}^{1+\delta}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ for all i , $\mathcal{T} \in \mathcal{C}^\delta(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ for some $0 < \delta < 1$, and where there exist positive constants a_0, b_0, b_1, c_0 and d_0 such that for all $\xi \in \mathbb{R}^n$ and all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$(84) \quad \xi^t \{ \nabla_p \mathcal{S}(x, z, p) \} \xi \geq c_0 |\xi|^2$$

$$(85) \quad |\nabla_p \mathcal{S}(x, z, p)| \leq d_0$$

$$(86) \quad (1 + |p|) |\partial_z \mathcal{S}| + |\nabla_x \mathcal{S}| + |\mathcal{T}| \leq d_0 (1 + |p|)^2$$

$$(87) \quad p \cdot \mathcal{S}(x, z, p) \geq a_0 |p|^2$$

$$(88) \quad \mathcal{T}(x, z, p) \operatorname{sign} z \leq b_0 |p| + b_1.$$

Then for any function $\varphi \in \mathcal{C}^0(\partial\Omega)$, there exists a solution $w \in \mathcal{C}^{2+\delta}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ of the Dirichlet problem

$$\begin{aligned} \mathcal{M}(w) &= 0 & \text{in } \Omega \\ w &= \varphi & \text{on } \partial\Omega. \end{aligned}$$

Remark 6.2. Theorem 15.19 in [6] is established only for $a_0 = 1$. Its generalization to any positive constant a_0 is straightforward. See the proof of Theorem 10.9 in [6] for details.

To prove Theorem 2.17, we will apply Theorem 6.1 to a family of truncations of the operator \mathcal{Q} defined in (6). For each $M > 0$, let $\chi_M \in \mathcal{C}^\infty(\mathbb{R})$ satisfy

$$(89) \quad \chi_M(z) = \begin{cases} z & \text{if } |z| \leq M \\ \frac{3}{2}M & \text{if } z \geq 2M \\ -\frac{3}{2}M & \text{if } z < -2M \end{cases}, \quad \left| \frac{d}{dz} \chi_M(z) \right| \leq 1.$$

Define

$$\mathcal{A}^M(x, z) = \mathcal{A}(x, \chi_M(z)), \quad \tilde{\gamma}^M(x, z) = \tilde{\gamma}(x, \chi_M(z)), \quad f^M(x, z) = f(x, \chi_M(z)),$$

and set

$$\mathcal{Q}^M w(x) = \operatorname{div} \mathcal{A}^M(x, w(x)) \nabla w(x) + \tilde{\gamma}^M(x, w(x)) \cdot \nabla w(x) + f^M(x, w(x)).$$

Note that if $|w(x)| \leq M$ then $\mathcal{Q}^M w(x) = \mathcal{Q}w(x)$.

Proposition 6.3. *For each $\varepsilon, M > 0$, the operators $\mathcal{Q}_\varepsilon^M = \mathcal{Q}^M + \varepsilon \Delta$ satisfy the hypothesis of Theorem 6.1 with*

$$\begin{aligned} a_0 &= c_0 = \varepsilon \\ b_0 &= \|\vec{\gamma}\|_{L^\infty(\tilde{\Gamma})}, \quad b_1 = \|f\|_{L^\infty(\tilde{\Gamma})} \\ d_0 &= \|\nabla_{(x,z)} \mathcal{A}\|_{L^\infty(\tilde{\Gamma})} + \|\vec{\gamma}\|_{L^\infty(\tilde{\Gamma})} + \|f\|_{L^\infty(\tilde{\Gamma})} + \varepsilon, \end{aligned}$$

where $\tilde{\Gamma} = \Omega \times [-\frac{3}{2}M, \frac{3}{2}M]$. Moreover, since \mathcal{A}^M , $\vec{\gamma}^M$ and f^M are smooth functions, the value of δ in Theorem 6.1 can be any value $0 < \delta < 1$.

Proof. With the notation of Theorem 6.1, and $\mathcal{A}(x, z) = \{a^{ij}(x, z)\}_{i,j=1}^n$, we have

$$\mathcal{Q}_\varepsilon^M(w) = \mathcal{M}(w) = \operatorname{div} \mathcal{S}(x, w, \nabla w) + \mathcal{T}(x, w, \nabla w)$$

with

$$(90) \quad \mathcal{S}^i(x, z, p) = \sum_{j=1}^n (a^{ij}(x, \chi_M(z)) + \varepsilon \delta_{ij}) p_j, \quad i = 1, \dots, n,$$

$$(91) \quad \mathcal{T}(x, z, p) = \vec{\gamma}(x, \chi_M(z)) \cdot p + f(x, \chi_M(z)).$$

Here $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Let us now verify (85)–(88).

(84). By (90),

$$\begin{aligned} \xi^t \{\nabla_p \mathcal{S}(x, z, p)\} \xi &= \sum_{i,\ell=1}^n \partial_{p_\ell} \left(\sum_{j=1}^n (a^{ij}(x, \chi_M(z)) + \varepsilon \delta_{ij}) p_j \right) \xi_\ell \xi_i \\ &= \sum_{i,\ell=1}^n (a^{i\ell}(x, \chi_M(z)) + \varepsilon \delta_{i\ell}) \xi_\ell \xi_i \\ &= \xi^t \mathcal{A}(x, \chi_M(z)) \xi + \varepsilon |\xi|^2 \\ &\geq \varepsilon |\xi|^2. \end{aligned}$$

Hence, (84) holds with $c_0 = \varepsilon$.

(85). From (90),

$$\begin{aligned} |\partial_{p_j} \mathcal{S}^i(x, z, p)| &= |a^{ij}(x, \chi_M(z)) + \varepsilon \delta_{ij}| \\ &\leq |a^{ij}(x, \chi_M(z))| + \varepsilon \\ &\leq \|\mathcal{A}\|_{L^\infty(\tilde{\Gamma})} + \varepsilon, \end{aligned}$$

where $\tilde{\Gamma} = \Omega \times [-\frac{3}{2}M, \frac{3}{2}M]$. Thus, (85) holds with $d_0 = \|\mathcal{A}\|_{L^\infty(\tilde{\Gamma})} + \varepsilon$.

(86). From (90) and (91),

$$\begin{aligned} &(1 + |p|) |\partial_z \mathcal{S}| + |\nabla_x \mathcal{S}| + |\mathcal{T}| \\ &= (1 + |p|) \left| \frac{d}{dz} \chi_M(z) \right| |\mathcal{A}_z(x, \chi_M(z)) p| + |\nabla_x \mathcal{A}(x, \chi_M(z)) p| \\ &\quad + |\vec{\gamma}(x, \chi_M(z)) \cdot p + f(x, \chi_M(z))| \\ &\leq \left(\|\nabla_{(x,z)} \mathcal{A}\|_{L^\infty(\tilde{\Gamma})} + \|\vec{\gamma}\|_{L^\infty(\tilde{\Gamma})} + \|f\|_{L^\infty(\tilde{\Gamma})} \right) (1 + |p|)^2, \end{aligned}$$

where we used that $|\frac{d}{dz} \chi_M(z)| \leq 1$. Thus (86) holds with

$$d_0 = \|\nabla_{(x,z)} \mathcal{A}\|_{L^\infty(\tilde{\Gamma})} + \|\vec{\gamma}\|_{L^\infty(\tilde{\Gamma})} + \|f\|_{L^\infty(\tilde{\Gamma})}.$$

(87). By (90) it follows that

$$p \cdot \mathcal{S}(x, z, p) = p \cdot (\mathcal{A}(x, \chi_M(z)) + \mathbf{I}\varepsilon) p \geq \varepsilon |p|^2.$$

Thus (87) holds with $a_0 = \varepsilon$.

(88). By (91),

$$\begin{aligned} \mathcal{T}(x, z, p) \operatorname{sign} z &= (\vec{\gamma}(x, \chi_M(z)) \cdot p + f(x, \chi_M(z))) \operatorname{sign} z \\ &\leq \|\vec{\gamma}\|_{L^\infty(\bar{\Gamma})} |p| + \|f\|_{L^\infty(\bar{\Gamma})}. \end{aligned}$$

Hence (88) holds with $b_0 = \|\vec{\gamma}\|_{L^\infty(\bar{\Gamma})}$ and $b_1 = \|f\|_{L^\infty(\bar{\Gamma})}$. \square

6.1. Proof of Theorem 2.17. Let $\Omega \Subset \tilde{\Omega}$ be a strongly convex domain as in the hypotheses of Theorem 2.17. Then Ω satisfies an exterior sphere condition at each point of $\partial\Omega$. Given a continuous function φ on $\partial\Omega$, let $M_0 = \sup_{\partial\Omega} |\varphi|$. By Theorem 6.1 and Proposition 6.3 with $M = M_0$, for all $\varepsilon > 0$ there exists $w^\varepsilon \in \mathcal{C}^{2+\delta}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$, $0 < \delta < 1$, such that w^ε is a solution of the Dirichlet problem

$$\begin{cases} \mathcal{Q}_\varepsilon^{M_0} w &= 0 & \text{in } \Omega \\ w &= \varphi & \text{on } \partial\Omega. \end{cases}$$

The solution w^ε also depends on M_0 , which is fixed. The smoothness assumptions on \mathcal{A} , $\vec{\gamma}$ and f and a standard bootstrapping argument (see, e.g., Theorems 6.2 and 6.3 in [2]) imply that $w^\varepsilon \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$. Since the operators $\mathcal{Q}_\varepsilon^{M_0}$ satisfy the hypotheses of Theorem 7.4 in the Appendix (the maximum principle), we have

$$(92) \quad \|w^\varepsilon\|_{L^\infty(\Omega)} \leq \|w^\varepsilon\|_{L^\infty(\partial\Omega)} = \|\varphi\|_{L^\infty(\partial\Omega)} = M_0.$$

Thus the functions w^ε are uniformly bounded by M_0 in $\bar{\Omega}$ for all $\varepsilon > 0$. From the definition of $\mathcal{Q}_\varepsilon^{M_0}$ it follows that $\mathcal{Q}w^\varepsilon + \varepsilon \Delta w^\varepsilon = \mathcal{Q}_\varepsilon^{M_0} w^\varepsilon = 0$ in Ω .

By (9), the coefficients of $\mathcal{Q} + \varepsilon \mathbf{I}$, namely the entries of $\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon \mathbf{I}$, satisfy

$$\sum_{i=1}^n (k^i(x, z) + \varepsilon) \xi_i^2 \leq \xi^t \mathcal{A}_\varepsilon(x, z) \xi \leq \Lambda \sum_{i=1}^n (k^i(x, z) + \varepsilon) \xi_i^2$$

for all $\xi \in \mathbb{R}^n$ and $(x, z) \in \Gamma = \Omega \times \mathbb{R}$. That is, \mathcal{A}_ε satisfies the diagonal condition for diagonal entries $k^i + \varepsilon$. Next, by (12),

$$\sum_{i=1}^n |\partial_i \mathcal{A}_\varepsilon(x, z) \xi|^2 + |\partial_z \mathcal{A}_\varepsilon(x, z) \xi|^2 \leq B_{\mathcal{A}}^2 \xi^t \mathcal{A}_\varepsilon \xi \leq B_{\mathcal{A}}^2 \xi^t \mathcal{A}_\varepsilon \xi$$

for all $\xi \in \mathbb{R}^n$, $(x, z) \in \Gamma'_{M_0}$. Hence \mathcal{A}_ε is subordinate, with the same constant $B_{\mathcal{A}}$ as for \mathcal{A} in Γ'_{M_0} . We also have by (16) and (17) that

$$\begin{aligned} |\partial_z \mathcal{A}_\varepsilon(x, z) \xi|^2 &= |\partial_z \mathcal{A}(x, z) \xi|^2 \leq B_{\mathcal{A}}^2 k^*(x, z) \xi^t \mathcal{A}(x, z) \xi \\ &\leq C B_{\mathcal{A}}^2 (k^*(x, z) + \varepsilon) \xi^t \mathcal{A}_\varepsilon(x, z) \xi \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^n |\partial_i \partial_z \mathcal{A}_\varepsilon(x, z) \xi|^2 + |\partial_z^2 \mathcal{A}_\varepsilon(x, z) \xi|^2 &\leq (B'_{\mathcal{A}})^2 \xi^t \mathcal{A}(x, z) \xi \\ &\leq (B'_{\mathcal{A}})^2 \xi^t \mathcal{A}_\varepsilon(x, z) \xi \end{aligned}$$

for all $\xi \in \mathbb{R}^n$, $(x, z) \in \Gamma'_{M_0}$.

Thus \mathcal{A}_ε satisfies the super subordination condition with the same constants as \mathcal{A} . Hence $\mathcal{Q} + \varepsilon$ satisfies the hypotheses of Theorem 2.13 uniformly in ε , $0 \leq \varepsilon \leq 1$. Applying Theorem 2.13 to the solutions w^ε , it follows that for any multi-index $\vec{\alpha}$ of nonnegative integers, the family $\{D^{\vec{\alpha}} w^\varepsilon, 0 < \varepsilon \leq 1\}$ is equicontinuous and uniformly bounded in any subdomain $\Omega' \Subset \Omega$. By the Arzela-Ascoli theorem, there is a subsequence $\{w^{\varepsilon_i}\}$ with $\varepsilon_i \rightarrow 0$ which converges in $\mathcal{C}^\infty(\Omega)$ to a function $w^0 \in \mathcal{C}^\infty(\Omega)$, i.e., $D^{\vec{\alpha}} w^{\varepsilon_i}$ converges to $D^{\vec{\alpha}} w^0$ uniformly on compact subsets of Ω , for all multi-indexes $\vec{\alpha}$. We will show that w^0 is a solution of the Dirichlet problem (25).

Since w^0 and all its derivatives are uniform limits of w^{ε_i} and their corresponding derivatives in compact subsets of Ω , then $|\Delta w^0| < \infty$ in Ω , and for all $x \in \Omega$,

$$\begin{aligned} \mathcal{Q}w^0(x) &= \operatorname{div} \mathcal{A}(x, w^0) \nabla w^0 + \vec{\gamma}(x, w^0) \cdot \nabla w^0 + f(x, w^0) \\ &= \lim_{i \rightarrow \infty} \operatorname{div} \mathcal{A}(x, w^{\varepsilon_i}) \nabla w^{\varepsilon_i} + \vec{\gamma}(x, w^{\varepsilon_i}) \cdot \nabla w^{\varepsilon_i} + f(x, w^{\varepsilon_i}) \\ &= \lim_{i \rightarrow \infty} \mathcal{Q}_{\varepsilon_i}^{M_0} w^{\varepsilon_i}(x) - \varepsilon_i \Delta w^{\varepsilon_i}(x) \\ &= -\varepsilon_i \lim_{i \rightarrow \infty} \Delta w^{\varepsilon_i}(x) = 0. \end{aligned}$$

Therefore $w^0 \in \mathcal{C}^\infty(\Omega)$ is a strong solution of the differential equation in the Dirichlet problem (25). Define $w^0 = \varphi$ on $\partial\Omega$ and recall that $w^\varepsilon = \varphi$ on $\partial\Omega$ if $\varepsilon > 0$. To finish the proof of the theorem we must check that $w^0 \in \mathcal{C}^0(\overline{\Omega})$.

Let $\omega(r)$ be the modulus of continuity of φ on $\partial\Omega$:

$$\omega(r) = \sup_{x, y \in \partial\Omega, |x-y| \leq r} |\varphi(x) - \varphi(y)|.$$

Then ω is continuous and $\omega(0) = 0$. By Lemma 4.10, taking a bigger ω if necessary, we may assume that ω is also concave and strictly increasing in $[0, \operatorname{diam} \Omega]$, and \mathcal{C}^2 in $(0, \operatorname{diam} \Omega]$.

By our hypothesis on γ , there exists $\eta_0 > 0$ such that $\gamma(x, z) = 0$ if $x \in \Omega$, $\operatorname{dist}(x, \partial\Omega) < \eta_0$ and $|z| \leq M_0$. Let x_0 be an arbitrary point on $\partial\Omega$, and let $h(x)$ be the barrier function for ω at x_0 provided by Lemma 7.6 in the Appendix, with Φ and Ω there chosen to be Ω and $\tilde{\Omega}$ respectively, and with $\nu = 2M_0$, $m_0 = 2M_0$, $\eta = \eta_0$ and $K = \|f\|_{L^\infty(\tilde{\Gamma})}$, where $\tilde{\Gamma} = \overline{\Omega} \times [-2M_0, 2M_0]$. Thus there is a neighborhood \mathcal{N} of x_0 with $\mathcal{N} \subset \{|x - x_0| < \eta_0\}$ and a function $h \in \mathcal{C}^\infty(\mathcal{N}) \cap \mathcal{C}^0(\overline{\mathcal{N}})$ such that

$$(93) \quad h(x) \leq -\omega(|x - x_0|),$$

$$(94) \quad \operatorname{div} \mathcal{A}(x, h(x) + m) \nabla h \geq \|f\|_{L^\infty(\tilde{\Gamma})},$$

$$(95) \quad \Delta h = \sum_{i=1}^n \partial_i^2 h > 0$$

for all $x \in \Omega \cap \mathcal{N}$ and $|m| \leq 2M_0$. Moreover,

$$(96) \quad h(x) \leq -2M_0 \quad \text{if } x \in \partial\mathcal{N} \cap \Omega,$$

$$(97) \quad h(x_0) = 0.$$

Now, by (93) and the continuity of h on $\overline{\mathcal{N}}$,

$$(98) \quad h(x) \leq -\omega(|x - x_0|) \leq \varphi(x) - \varphi(x_0) = w^\varepsilon(x) - \varphi(x_0) \quad \text{if } x \in \overline{\mathcal{N}} \cap \partial\Omega, \varepsilon > 0.$$

By (96) and (92),

$$h(x) \leq -2M_0 \leq w^\varepsilon(x) - \varphi(x_0), \quad \text{if } x \in \partial\mathcal{N} \cap \Omega, \varepsilon > 0.$$

Therefore,

$$(99) \quad h(x) + \varphi(x_0) \leq w^\varepsilon(x) \quad \text{if } x \in \partial\mathcal{N}, \varepsilon > 0.$$

On the other hand, since w^ε is a solution of $\mathcal{Q}_\varepsilon^{M_0} w^\varepsilon = 0$ and $\mathcal{N} \subset \{|x - x_0| < \eta_0\}$,

$$\operatorname{div} \mathcal{A}^\varepsilon(x, w^\varepsilon) \nabla w^\varepsilon = -f(x, w^\varepsilon) \leq \|f\|_{L^\infty(\tilde{\Gamma})} \quad \text{in } \mathcal{N} \cap \Omega,$$

where the last inequality follows from (92). Thus, letting \mathcal{L}_ε be the quasilinear operator $\mathcal{L}_\varepsilon = \operatorname{div} \mathcal{A}^\varepsilon(x, \cdot) \nabla$, we have by (94) and (99) that

$$\begin{aligned} \mathcal{L}_\varepsilon(h + \varphi(x_0)) &\geq \mathcal{L}_\varepsilon w^\varepsilon && \text{in } \mathcal{N} \cap \Omega, \\ h(x) + \varphi(x_0) &< w^\varepsilon(x) && \text{if } x \in \partial(\mathcal{N} \cap \Omega). \end{aligned}$$

From the comparison principle Lemma 7.5 applied to \mathcal{L}_ε and the functions $w^\varepsilon, h + \varphi(x_0)$ in $\mathcal{N} \cap \Omega$, we obtain

$$(100) \quad h(x) \leq w^\varepsilon(x) - \varphi(x_0) \quad \text{if } x \in \mathcal{N} \cap \Omega.$$

Since h is continuous in $\overline{\mathcal{N}}$ and $h(x_0) = 0$, given any $\sigma > 0$, there exists $\delta_0 > 0$ independent of ε such that

$$-\sigma < w^\varepsilon(x) - \varphi(x_0) \quad \text{if } x \in \mathcal{N} \cap \{|x - x_0| < \delta_0\} \cap \Omega, \quad \varepsilon > 0.$$

Proceeding in a similar fashion for the function $\varphi(x_0) - w^\varepsilon(x)$, we obtain

$$(101) \quad |w^\varepsilon(x) - \varphi(x_0)| < \sigma \quad \text{if } x \in \mathcal{N} \cap \{|x - x_0| < \delta_0\} \cap \Omega, \quad \varepsilon > 0.$$

Let us now show that w^0 is continuous on $\overline{\Omega}$. Suppose not, and let x_0 be a point of discontinuity of w^0 in $\overline{\Omega}$. Since w^0 is smooth in Ω , x_0 must lie on $\partial\Omega$. Since $w^0 = \varphi$ on $\partial\Omega$ and φ is continuous by hypothesis, there exist points $\{x_k\}_{k=1}^\infty \subset \Omega$ and $\tilde{\sigma} > 0$ such that $x_k \rightarrow x_0$ and $|w^0(x_k) - \varphi(x_0)| \geq \tilde{\sigma}$ for all k . By (101) with $\sigma = \tilde{\sigma}/2$, there exists $\delta > 0$ independent of ε such that $|w^\varepsilon(x) - \varphi(x_0)| \leq \tilde{\sigma}/2$ if $x \in \Omega$ and $|x - x_0| < \delta$. Choose $x = x_{k_0}$ for k_0 so large that $|x_{k_0} - x_0| < \delta$. Then $|w^\varepsilon(x_{k_0}) - \varphi(x_0)| \leq \tilde{\sigma}/2$ for all ε . However, $w^\varepsilon(x_{k_0}) \rightarrow w^0(x_{k_0})$ as $\varepsilon \rightarrow 0$, so $|w^0(x_{k_0}) - \varphi(x_0)| \leq \tilde{\sigma}/2$, which is a contradiction. Hence $w^0 \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and w^0 is a solution to the Dirichlet problem (25).

When $\gamma \equiv 0$, uniqueness follows by Lemma 7.5 in the Appendix. This finishes the proof of Theorem 2.17.

6.2. Proof of Theorem 2.18. Under the hypotheses of Theorem 2.18, let w be a continuous weak solution of

$$\operatorname{div} \mathcal{A}(x, w) \nabla w + f(x, w) = 0 \quad \text{in } \Omega.$$

Given $\bar{x} \in \Omega$, let Φ be the ball centered at \bar{x} with radius $r = \frac{1}{2} \operatorname{dist}(\bar{x}, \partial\Omega)$. Then Φ is strongly convex. By Theorem 2.17, there is a continuous strong solution u of the Dirichlet problem

$$\begin{cases} \mathcal{Q}u &= 0 & \text{in } \Phi \\ u &= w & \text{on } \partial\Phi. \end{cases}$$

Moreover, $u \in C^0(\overline{\Phi}) \cap C^\infty(\Phi)$. By restricting \bar{x} to a compact set $\Omega' \subset \Omega$, the convex character λ_0 of Φ is bounded below away from zero, the bound depending on $\operatorname{dist}(\Omega', \partial\Omega)$, and hence the constants \mathcal{C}_N controlling the derivatives are independent of λ_0 . By the uniqueness part of the comparison principle, Lemma 7.5, it follows that $u = w$ in Φ and therefore w is smooth inside Φ with control on all its derivatives in compact subsets of Φ . This finishes the proof of Theorem 2.18.

7. APPENDIX

This Appendix is divided into four subsections in which we give some technical details about facts that we used earlier: degenerate Sobolev spaces and weak solutions, a maximum principle, a comparison principle, and barriers for the Dirichlet problem.

7.1. Degenerate Sobolev Spaces and Weak Solutions.

7.1.1. The weak degenerate Sobolev space $H_{\mathcal{X}}^{1,2}(\Omega)$. We first describe the degenerate Sobolev spaces used in the paper, beginning with a standard definition.

Definition 7.1 (Weak X derivative). *Let X be a locally Lipschitz vector field on Ω , i.e., $X = \mathbf{v} \cdot \nabla$ with $\mathbf{v} \in \operatorname{Lip}_{\text{loc}}(\Omega)$, the class of locally Lipschitz continuous \mathbb{R}^n -valued functions on Ω . X is initially defined on real-valued functions $w \in \operatorname{Lip}_{\text{loc}}(\Omega)$ by $Xw = \mathbf{v} \cdot \nabla w$. We say that a locally integrable function g is the weak derivative Xw of a locally integrable function w if*

$$(102) \quad \int_{\Omega} g \varphi = - \int_{\Omega} w X' \varphi = - \int_{\Omega} w \nabla \cdot (\mathbf{v} \varphi) \quad \text{for all } \varphi \in \operatorname{Lip}_0(\Omega).$$

The weak derivative Xw is clearly unique if it exists, and Xw exists and coincides with $\mathbf{v} \cdot \nabla w$ if $w \in \text{Lip}_{\text{loc}}(\Omega)$.

Definition 7.2 (Weak degenerate Sobolev space). *Let $\mathcal{X} = \{X_j\}_{j=1}^m$ where $X_j = \mathbf{v}_j$ are $\text{Lip}_{\text{loc}}(\Omega)$ vector fields on $\Omega \subset \mathbb{R}^n$. The degenerate Sobolev space $H_{\mathcal{X}}^{1,2}(\Omega)$ is defined as the inner product space consisting of all $w \in L^2(\Omega)$ whose weak derivatives $X_j w$ are also in $L^2(\Omega)$. The inner product in $H_{\mathcal{X}}^{1,2}(\Omega)$ is defined by*

$$(103) \quad \langle w, v \rangle_{\mathcal{X}} = \int_{\Omega} wv \, dx + \int_{\Omega} \mathcal{X}w \cdot \mathcal{X}v \, dx,$$

where we denote $\mathcal{X}w = (X_1 w, \dots, X_m w)$, and the norm is

$$\|w\|_{H_{\mathcal{X}}^{1,2}(\Omega)} = \langle w, w \rangle_{\mathcal{X}}^{1/2} = \left(\|w\|_{L^2(\Omega)}^2 + \|\mathcal{X}w\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

We now define what we mean by ∇w if $w \in H_{\mathcal{X}}^{1,2}(\Omega)$ for a collection $\mathcal{X} = \{X_j\}_{j=1}^m = \{\mathbf{v}_j \cdot \nabla\}_{j=1}^m$ of $\text{Lip}_{\text{loc}}(\Omega)$ vector fields. For such w and \mathcal{X} , there is a sequence $\{w_k\}_{k=1}^{\infty}$ of $\text{Lip}(\Omega)$ functions and a vector $\vec{W}(x) \in \mathbb{R}^n$ satisfying $\mathbf{v}_j \cdot \vec{W} \in L^2(\Omega)$ for all j and

$$(104) \quad \|w_k - w\|_{L^2(\Omega)} + \sum_j \|X_j w_k - \mathbf{v}_j \cdot \vec{W}\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This is proved in [12] (see also [4], [5]) in case all $\mathbf{v}_j \in \text{Lip}(\Omega)$ but remains true if all $\mathbf{v}_j \in \text{Lip}_{\text{loc}}(\Omega)$ by examining the proof in [12].

Then

$$(105) \quad \mathcal{X}w = (X_1 w, \dots, X_m w) = (\mathbf{v}_1 \cdot \vec{W}, \dots, \mathbf{v}_m \cdot \vec{W})$$

since for all $\varphi \in \text{Lip}_0(\Omega)$,

$$\begin{aligned} \int_{\Omega} w \nabla \cdot (\mathbf{v}_j \varphi) &= \lim_{k \rightarrow \infty} \int_{\Omega} w_k \nabla \cdot (\mathbf{v}_j \varphi) \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} (\mathbf{v}_j \cdot \nabla w_k) \varphi = \lim_{k \rightarrow \infty} \int_{\Omega} (X_j w_k) \varphi \\ &= - \int_{\Omega} (\mathbf{v}_j \cdot \vec{W}) \varphi. \end{aligned}$$

Moreover, if $\{w'_k\}$ and \vec{W}' are another such sequence and vector for the same w , it follows similarly that $\int_{\Omega} w \nabla \cdot (\mathbf{v}_j \varphi) = - \int_{\Omega} (\mathbf{v}_j \cdot \vec{W}') \varphi$. Hence

$$\int_{\Omega} (\mathbf{v}_j \cdot \vec{W}) \varphi = \int_{\Omega} (\mathbf{v}_j \cdot \vec{W}') \varphi$$

for all $\varphi \in \text{Lip}_0(\Omega)$, so that

$$(106) \quad \mathbf{v}_j \cdot \vec{W} = \mathbf{v}_j \cdot \vec{W}' \quad \text{for all } j.$$

In this sense, \vec{W} is unique, i.e., \vec{W} is uniquely determined by w up to its dot product with each vector \mathbf{v}_j . We will often abuse notation by writing $\vec{W} = \nabla w$. Any particular \vec{W} as above will be called a *representative* of ∇w . Then $X_j w = \mathbf{v}_j \cdot \nabla w$, $j = 1, \dots, m$, for all $w \in H_{\mathcal{X}}^{1,2}(\Omega)$. Furthermore, by (104), the sequence $\{w_k\}$ above satisfies

$$\|w_k - w\|_{H_{\mathcal{X}}^{1,2}(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Suppose that $\Omega' \subset \Omega$ and $M > 0$, and let $\mathcal{X} = \{X_j\} = \{\mathbf{v}_j \cdot \nabla\}$ and $H_{\mathcal{X},0}^{1,2}(\Omega)$ be as above. We claim that if $\mathcal{A}(x, z)$ and $\vec{\gamma}(x, z)$ satisfy

$$(107) \quad \xi \cdot \mathcal{A}(x, z) \xi \leq c \sum_j (\mathbf{v}_j(x) \cdot \xi)^2 \quad \text{and} \quad (\vec{\gamma}(x, z) \cdot \xi)^2 \leq c \sum_j (\mathbf{v}_j(x) \cdot \xi)^2$$

for all $(x, z, \xi) \in \Omega' \times (-M, M) \times \mathbb{R}^n$, then $\sqrt{\mathcal{A}(x, z)} \nabla w$ and $\vec{\gamma}(x, z) \cdot \nabla w$ are well-defined for any $(x, z) \in \Omega' \times (-M, M)$ and any $w \in H_{\mathcal{X}, 0}^{1,2}(\Omega)$, i.e., that if \vec{W} and \vec{W}' are any two representatives of ∇w , then $\sqrt{\mathcal{A}(x, z)} \vec{W}(x) = \sqrt{\mathcal{A}(x, z)} \vec{W}'(x)$ and $\vec{\gamma}(x, z) \cdot \vec{W}(x) = \vec{\gamma}(x, z) \cdot \vec{W}'(x)$. In fact, since $\mathbf{v}_j \cdot (\vec{W} - \vec{W}') = \mathbf{v}_j \cdot \vec{W} - \mathbf{v}_j \cdot \vec{W}' = 0$ for all j by (106), this follows immediately from (107) by choosing $\xi = \vec{W}(x) - \vec{W}'(x)$.

Let $\vec{k}(x, z) = (k^i(x, z))_{i=1, \dots, n}$ and $\mathcal{A}(x, z)$ be as in Theorem 2.17, that is, with $\Gamma = \tilde{\Omega} \times \mathbb{R}$,

- (i) $\vec{k}(x, z) \in \mathcal{C}^2(\Gamma)$ and satisfies the nondegeneracy Condition 2.3 in Γ ,
- (ii) $\mathcal{A} \in \mathcal{C}^\infty(\Gamma)$ and satisfies the diagonal Condition 2.5 in Γ ,
- (iii) \mathcal{A} satisfies the super subordination Condition 2.10 in Γ .

The particular vector fields that we will use are

$$(108) \quad \partial_1, \sqrt{k^2(x, 0)} \partial_2, \dots, \sqrt{k^n(x, 0)} \partial_n.$$

We claim that since $k^2(x, 0), \dots, k^n(x, 0) \in \mathcal{C}^2(\Omega)$ and are nonnegative, the Wirtinger inequality (11) implies that the vector fields (108) belong to $\text{Lip}_{\text{loc}}(\Omega)$. To see why, fix i and denote $k^i(x, 0) = k(x)$. For a Euclidean ball $D \Subset \Omega$, $\varepsilon > 0$ and all $x_1, x_2 \in D$, we have

$$\begin{aligned} \left| \sqrt{k(x_1) + \varepsilon} - \sqrt{k(x_2) + \varepsilon} \right| &\leq \left\| \nabla \sqrt{k + \varepsilon} \right\|_{L^\infty(D)} |x_1 - x_2| \\ &= \left\| \frac{\nabla k}{\sqrt{k + \varepsilon}} \right\|_{L^\infty(D)} |x_1 - x_2| \\ &\leq C_D \left\| \frac{\sqrt{k}}{\sqrt{k + \varepsilon}} \right\|_{L^\infty(D)} |x_1 - x_2| \quad \text{by (11),} \end{aligned}$$

where C_D depends on k and $\text{dist}(D, \partial\Omega)$. Hence

$$\left| \sqrt{k(x_1) + \varepsilon} - \sqrt{k(x_2) + \varepsilon} \right| \leq C_D |x_1 - x_2|, \quad x_1, x_2 \in D,$$

uniformly in ε . Letting $\varepsilon \rightarrow 0$ and using the Heine-Borel theorem to cover any compact subset of Ω by a finite number of balls proves our claim.

7.1.2. \mathcal{X} -weak solutions of quasilinear equations. Here we make precise the notion of a “weak solution” of the quasilinear equation (6). For $k^i(x, z)$ as in the hypotheses of our main results Theorems 2.18 and 2.17, we let $\mathcal{X} = \{X_j\}_{j=1}^n$ with $X_j = \sqrt{k^i(x, 0)} \frac{\partial}{\partial x_j}$. An analogous definition can be given for any collection of locally Lipschitz vector fields.

Definition 7.3. A function $w \in H_{\mathcal{X}}^{1,2}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$ is a weak solution of

$$(109) \quad \mathcal{Q}w = \text{div } \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) = 0 \quad \text{in } \Omega$$

if for all $u \in \text{Lip}_0(\Omega)$,

$$(110) \quad \int_{\Omega} (\nabla u)^t \mathcal{A}(x, w) \nabla w \, dx = \int_{\Omega} u \vec{\gamma}(x, w) \cdot \nabla w \, dx + \int_{\Omega} u f(x, w) \, dx.$$

Given $w_0, w_1 \in H_{\mathcal{X}}^{1,2}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$, we say that $\mathcal{Q}w_1 \geq \mathcal{Q}w_0$ in Ω if

$$\begin{aligned} &\int_{\Omega} (\nabla u)^t \mathcal{A}(x, w_1) \nabla w_1 \, dx - \int_{\Omega} u \vec{\gamma}(x, w_1) \cdot \nabla w_1 \, dx - \int_{\Omega} u f(x, w_1) \, dx \\ &\leq \int_{\Omega} (\nabla u)^t \mathcal{A}(x, w_0) \nabla w_0 \, dx - \int_{\Omega} u \vec{\gamma}(x, w_0) \cdot \nabla w_0 \, dx - \int_{\Omega} u f(x, w_0) \, dx \end{aligned}$$

for all $u \in \text{Lip}_0(\Omega)$.

To show that the integrals in (110) converge absolutely, note that if $w \in L_{\text{loc}}^\infty(\Omega)$, then by Lemma 2.15 and the diagonal condition (9), we have that for all $x \in \Omega' \Subset \Omega$,

$$\xi^t \mathcal{A}(x, w(x)) \xi \approx \xi^t \mathcal{A}(x, 0) \xi \approx \sum_{i=1}^n k^i(x, 0) \xi_i^2,$$

with constants which depend on \mathcal{A} , Ω' and $M_0 = \|w\|_{L^\infty(\Omega')}$. Hence, since $w \in H_{\mathcal{X}}^{1,2}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$, $u \in \text{Lip}_0(\Omega)$, and $f(x, z)$ is continuous, it follows that

$$\nabla_{\sqrt{\mathcal{A}}, w} w, \nabla_{\sqrt{\mathcal{A}}, w} u, \vec{\gamma}(x, w(x)) \cdot \nabla w, -f(x, w(x)) \in L_{\text{loc}}^2(\Omega).$$

Consequently, each integral in (110) converges absolutely, and the same is true for the integrals in Definition 7.3.

Alternately, we can make sense of weak solutions $w \in H_{\mathcal{X}}^{1,2}(\Omega)$ which are not necessarily locally bounded by assuming that the coefficient matrix is bounded in z locally in x , that $\vec{\gamma}(x, z)$ is of subunit type globally in z locally in x , and that $\sup_z |f(x, z)|$ is locally integrable.

7.2. A Maximum Principle. We will now prove the following result.

Theorem 7.4. *Let \mathcal{A} satisfy (9), $\vec{\gamma}$ be of subunit type with respect to \mathcal{A} in $\Omega \times \mathbb{R}$, and f be a continuous function on $\Omega \times \mathbb{R}^n$ which satisfies $f(x, 0) = 0$ for $x \in \Omega$ and*

$$(111) \quad f(x, z) \text{ sign } z \leq 0 \quad \text{in } \Omega \times \mathbb{R},$$

$$(112) \quad f(x, z_1) - f(x, z_2) \leq 0 \quad \text{if } x \in \Omega \text{ and } z_1 \geq z_2.$$

If w is a smooth function in Ω which is continuous on $\overline{\Omega}$ and satisfies

$$(113) \quad \text{div } \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) \geq 0 \quad \text{in } \Omega$$

in the weak sense, i.e., satisfies

$$(114) \quad \int \nabla \varphi \cdot \mathcal{A}(x, w) \nabla w \leq \int \varphi \vec{\gamma}(x, w) \cdot \nabla w + \int \varphi f(x, w)$$

for all nonnegative $\varphi \in \text{Lip}_0(\Omega)$, then

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w^+.$$

On the other hand, if the opposite inequality holds in (113), i.e., if

$$(115) \quad \text{div } \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) \leq 0 \quad \text{in } \Omega$$

in the weak sense, then

$$(116) \quad \inf_{\Omega} w \geq \inf_{\partial\Omega} -(w^-) \quad \left(= -\sup_{\partial\Omega} w^- \right),$$

where $w^- := -w$ if $w \leq 0$ and $w^- := 0$ if $w > 0$. In particular, if w is a weak solution of $\text{div } \mathcal{A}(x, w) \nabla w + \vec{\gamma}(x, w) \cdot \nabla w + f(x, w) = 0$ in Ω , then $\sup_{\Omega} |w| \leq \sup_{\partial\Omega} |w|$.

Proof. Assume first that w satisfies (113), and recall that w is smooth by assumption. Let $\omega_0 = \sup_{\partial\Omega} w^+$ and

$$v_\tau(x) = (w(x) - \omega_0 - \tau)^+, \quad \tau > 0, x \in \Omega.$$

Then v_τ is nonnegative and Lipschitz continuous in Ω , and v_τ has compact support in Ω since w is continuous on $\overline{\Omega}$ by hypothesis and $\tau > 0$. Also, for any x , $v_\tau(x) > 0$ if and only if $w(x) > \omega_0 + \tau$. Let $\Phi_\tau = \{x \in \Omega : v_\tau(x) > 0\}$. Then $v_\tau = \chi_{\Phi_\tau}(w - \omega_0 - \tau)$ on Ω and $\nabla v_\tau = \chi_{\Phi_\tau} \nabla w$ a.e. on Ω . By choosing $\varphi = v_\tau$ in (114), we obtain

$$\int \nabla v_\tau \cdot \mathcal{A}(x, w) \nabla w \leq \int v_\tau \vec{\gamma}(x, w) \cdot \nabla w + \int v_\tau f(x, w).$$

In this inequality, the right-hand side satisfies

$$\int_{\Phi_\tau} v_\tau \tilde{\gamma}(x, w) \cdot \nabla w + \int_{\Phi_\tau} v_\tau f(x, w) = \int_{\Phi_\tau} v_\tau \tilde{\gamma}(x, w) \cdot \nabla v_\tau + \int_{\Phi_\tau} v_\tau f(x, w),$$

while the left-hand side is $\int_{\Phi_\tau} \nabla v_\tau \cdot \mathcal{A}(x, w) \nabla v_\tau$. Hence

$$(117) \quad \int_{\Phi_\tau} \nabla v_\tau \cdot \mathcal{A}(x, w) \nabla v_\tau \leq \int_{\Phi_\tau} v_\tau \tilde{\gamma}(x, w) \cdot \nabla v_\tau + \int_{\Phi_\tau} v_\tau f(x, w) = I + II.$$

We have

$$II = \int_{\Phi_\tau} v_\tau f(x, v_\tau) + \int_{\Phi_\tau} v_\tau [f(x, w) - f(x, v_\tau)] \leq 0 + 0 = 0$$

by (111) and (112) since $v_\tau > 0$ and $w > v_\tau$ on Φ_τ (note that $\omega \geq 0$). Recalling that w is assumed to be continuous on $\bar{\Omega}$ and so is bounded there, we may choose M with $M > w$ on Ω . Since $\tilde{\gamma}$ is of subunit type with respect to \mathcal{A} , Schwarz's inequality implies that

$$I \leq \frac{1}{4} \int_{\Phi_\tau} \nabla v_\tau \cdot \mathcal{A}(x, w) \nabla v_\tau + 4B_\gamma^2 \int_{\Phi_\tau} v_\tau^2,$$

where $B_\gamma = B_\gamma(\Omega_\tau, M)$. From (117) and the estimates for I and II , we obtain

$$(118) \quad \int_{\Phi_\tau} \nabla v_\tau \cdot \mathcal{A}(x, w) \nabla v_\tau \leq C^2 B_\gamma^2 \int_{\Phi_\tau} v_\tau^2.$$

By the one-dimensional Sobolev estimate,

$$\int_{\Phi_\tau} v_\tau^2 \leq C^2 R^2 \int_{\Phi_\tau} |\partial_1 v_\tau|^2, \quad R = \text{diam}(\Omega).$$

Combining this with (118) gives

$$\int_{\Phi_\tau} v_\tau^2 \leq C^2 R^2 B_\gamma^2 \int_{\Phi_\tau} v_\tau^2.$$

Thus, assuming that $R < (CB_\gamma)^{-1}$, we obtain $\int_{\Phi_\tau} v_\tau^2 = 0$. Then Φ_τ is empty, i.e., $v_\tau = 0$ on Ω and therefore $w \leq \omega_0 + \tau$ on Ω .

To drop the restriction that $R < (CB_\gamma(\Omega_\tau, M))^{-1}$, let $N = CB_\gamma \text{diam}(\Omega) + 1$ and

$$\tilde{\Omega} = \frac{\Omega}{N} = \left\{ \frac{x}{N} : x \in \Omega \right\}.$$

Also, for $x \in \tilde{\Omega}$, let

$$\begin{aligned} \tilde{w}(x) &= w(Nx), & \tilde{\mathcal{A}}(x, z) &= N^{-2} \mathcal{A}(Nx, z), \\ \tilde{\gamma}(x, z) &= N^{-1} \gamma(Nx, z), & \tilde{f}(x, z) &= f(Nx, z). \end{aligned}$$

Then if $x \in \tilde{\Omega}$,

$$\begin{aligned} & \text{div } \tilde{\mathcal{A}}(x, \tilde{w}) \nabla \tilde{w} + \tilde{\gamma}(x, \tilde{w}) \cdot \nabla \tilde{w} + \tilde{f}(x, \tilde{w}) \\ &= N^{-2} \text{div} [\mathcal{A}(Nx, w(Nx)) \nabla (w(Nx))] + N^{-1} \gamma(Nx, w(Nx)) \cdot \nabla (w(Nx)) \\ & \quad + f(Nx, w(Nx)) \\ &= \text{div } \mathcal{A}(y, w(y)) \nabla w(y) + \gamma(y, w(y)) \cdot \nabla w(y) + f(y, w(y)) \geq 0 \end{aligned}$$

by (113), where $y = Nx \in \Omega$. Thus \tilde{w} satisfies an analogue of (113) in $\tilde{\Omega}$. Moreover, since $\tilde{\gamma}$ is of subunit type with respect to \mathcal{A} in $\Omega \times \mathbb{R}$ (Definition 2.7), it follows that $\tilde{\gamma}$ is of subunit type with respect to $\tilde{\mathcal{A}}$ in $\tilde{\Omega} \times \mathbb{R}$ with constant $\tilde{B}_\gamma = B_\gamma$: indeed,

$$\begin{aligned} \left| \tilde{\gamma}(x, z) \cdot \xi \right|^2 &= \left| N^{-1} \gamma(Nx, z) \cdot \xi \right|^2 \leq N^{-2} B_\gamma^2 \xi^t \mathcal{A}(Nx, z) \xi \\ &= B_\gamma^2 \xi^t \tilde{\mathcal{A}}(x, z) \xi. \end{aligned}$$

Also,

$$\tilde{f}(x, z) \text{sign } z = f(Nx, z) \text{sign } z \leq 0 \quad \text{in } \tilde{\Omega} \times \mathbb{R},$$

$$\tilde{f}(x, z_1) - \tilde{f}(x, z_2) = f(Nx, z_1) - f(Nx, z_2) \leq 0 \quad \text{if } z_1 \geq z_2, x \in \tilde{\Omega}$$

and

$$\text{diam } \tilde{\Omega} = \frac{\text{diam } \Omega}{N} = \frac{\text{diam } \Omega}{CB_\gamma \text{diam } \Omega + 1} < \frac{1}{CB_\gamma} = \frac{1}{C\widehat{B}_\gamma}.$$

Then $\tilde{w} \leq \sup_{\partial\tilde{\Omega}}(\tilde{w})^+ + \tau$ in $\tilde{\Omega}$ for all $\tau > 0$, i.e., $w \leq \sup_{\partial\Omega} w^+ + \tau$ in Ω for all $\tau > 0$. Letting $\tau \rightarrow 0$, we obtain $w \leq \sup_{\partial\Omega} w^+$ as desired.

To prove (116), let (115) hold and define $\tilde{\mathcal{A}}(x, z) = \mathcal{A}(x, -z)$, $\tilde{\gamma}(x, z) = \gamma(x, -z)$ and $\tilde{f}(x, z) = -f(x, z)$ for $x \in \Omega$. Since \mathcal{A} satisfies (9) and γ is subunit with respect to \mathcal{A} in $\Omega \times \mathbb{R}$, it follows that $\tilde{\mathcal{A}}$ satisfies (9) and $\tilde{\gamma}$ is subunit with respect to $\tilde{\mathcal{A}}$ in $\Omega \times \mathbb{R}$. From (111) and (112) for f , we obtain (111) and (112) for \tilde{f} :

$$\begin{aligned} \tilde{f}(x, z) \text{sign } z &= -f(x, -z) \text{sign } z \\ &= f(x, -z) \text{sign}(-z) \leq 0 \quad \text{in } \Omega \times \mathbb{R}, \\ \tilde{f}(x, z_1) - \tilde{f}(x, z_2) &= f(x, -z_2) - f(x, -z_1) \leq 0 \quad \text{if } z_1 \geq z_2, x \in \Omega. \end{aligned}$$

Now let $\tilde{w}(x) = -w(x)$ and note that

$$\begin{aligned} \text{div } \tilde{\mathcal{A}}(x, \tilde{w}(x)) \nabla \tilde{w}(x) + \tilde{\gamma}(x, \tilde{w}(x)) \cdot \nabla \tilde{w}(x) + \tilde{f}(x, \tilde{w}(x)) &= \\ -[\text{div } \mathcal{A}(x, w(x)) \nabla w(x) + \gamma(x, w(x)) \cdot \nabla w(x) + f(x, w(x))] &\geq 0 \quad \text{in } \Omega. \end{aligned}$$

By the previous case, $\sup_{\Omega} \tilde{w} \leq \sup_{\partial\Omega} \tilde{w}^+$. Equivalently,

$$\sup_{x \in \Omega} (-w(x)) \leq \sup_{x \in \partial\Omega} (-w(x))^+, \quad \text{or} \quad \inf_{x \in \Omega} w(x) \geq \inf_{x \in \partial\Omega} -(w(x)^-),$$

which completes the proof of Theorem 7.4. \square

7.3. A Comparison Principle.

Lemma 7.5. *Suppose that $\mathcal{A}(x, z)$ satisfies (9) and (16), and that f is nonincreasing in z , i.e.,*

$$(119) \quad f_z(x, z) \leq 0, \quad (x, z) \in \Gamma.$$

Let $w_0, w_1 \in (H_{\mathcal{X}}^{1,2}(\Omega) \cup C^\infty(\Omega)) \cap C^0(\overline{\Omega})$ satisfy $w_0 + \kappa \geq w_1$ on $\partial\Omega$ for some constant $\kappa \geq 0$ and

$$(120) \quad \mathcal{P}(w_1) \geq \mathcal{P}(w_0) \quad \text{in } \Omega$$

(in the sense of Definition 7.3), where

$$\mathcal{P}(w) = \text{div } \mathcal{A}(x, w) \nabla w + f(x, w).$$

Then $w_0 + \kappa \geq w_1$ in Ω . In particular, if $\mathcal{P}w_0 = \mathcal{P}w_1$ in Ω and $w_0 = w_1$ on $\partial\Omega$, then $w_0 = w_1$ in $\overline{\Omega}$.

Proof. First we will assume that $w_0, w_1 \in H_{\mathcal{X}}^{1,2}(\Omega) \cap C^0(\overline{\Omega})$. Given $\tau > 0$, let $u_\tau = w_1 - w_0 - \kappa - \tau$ and $u_\tau^+ = \max\{u_\tau, 0\}$. Then u_τ^+ is a nonnegative continuous function compactly supported in Ω . Denote $K = \text{supp}(u_\tau^+)$ and $\delta_0 = \text{dist}(K, \partial\Omega)$. Let ψ_δ be a smooth approximation of the identity with $\delta > 0$, i.e., $\psi_\delta(x) = \delta^{-n} \psi(x/\delta)$ where $\psi \in C_0^\infty(B_1)$ and $\int \psi dx = 1$; here B_1 denotes the unit ball in \mathbb{R}^n . For $0 < \varepsilon < 1$ and $0 < \delta < \delta_0/2$, set

$$\varphi_{\tau, \varepsilon, \delta} = \frac{u_\tau^+}{u_\tau^+ + \varepsilon} * \psi_\delta.$$

Then $\varphi_{\tau, \varepsilon, \delta}$ is nonnegative, smooth and compactly supported in Ω . From (120),

$$\int_{\Omega} [\mathcal{A}(x, w_1) \nabla w_1 - \mathcal{A}(x, w_0) \nabla w_0] \nabla \varphi_{\tau, \varepsilon, \delta} - \int_{\Omega} [f(x, w_1) - f(x, w_0)] \varphi_{\tau, \varepsilon, \delta} \leq 0.$$

Since $w_0, w_1, \varphi_{\tau, \varepsilon, \delta} \in H_{\mathcal{X}}^{1,2}(\Omega)$ and $\varphi_{\tau, \varepsilon, \delta}$ is continuous and has compact support, all the integrals above are absolutely convergent. By Proposition 1.2.2 in [4], $\varphi_{\tau, \varepsilon, \delta} \rightarrow u_{\tau}^{+}/(u_{\tau}^{+} + \varepsilon)$ in $H_{\mathcal{X}}^{1,2}(\Omega'')$ as $\delta \rightarrow 0^{+}$ for any open $\Omega'' \Subset \Omega'$. Letting $\delta \rightarrow 0^{+}$, we obtain

$$(121) \quad \begin{aligned} & \varepsilon \int_{\Omega} [\mathcal{A}(x, w_1) \nabla w_1 - \mathcal{A}(x, w_0) \nabla w_0] \frac{\nabla u_{\tau}^{+}}{(u_{\tau}^{+} + \varepsilon)^2} \\ & - \int_{\Omega} [f(x, w_1) - f(x, w_0)] \frac{u_{\tau}^{+}}{u_{\tau}^{+} + \varepsilon} \leq 0, \end{aligned}$$

where we used that $\nabla(u_{\tau}^{+}/(u_{\tau}^{+} + \varepsilon)) = \varepsilon \nabla u_{\tau}^{+}/(u_{\tau}^{+} + \varepsilon)^2$.

Set $u = w_1 - w_0 - \kappa$ and $w_t = tw_1 + (1-t)w_0$, $0 \leq t \leq 1$. Then $\partial_t w_t = u + \kappa$, and by (119),

$$(122) \quad \begin{aligned} [f(x, w_1) - f(x, w_0)] \frac{u_{\tau}^{+}}{u_{\tau}^{+} + \varepsilon} &= \frac{u_{\tau}^{+}}{u_{\tau}^{+} + \varepsilon} \int_0^1 \partial_t [f(x, w_t)] dt \\ &= \frac{u_{\tau}^{+}(u + \kappa)}{u_{\tau}^{+} + \varepsilon} \int_0^1 f_z(x, w_t) dt \leq 0, \end{aligned}$$

where we used that $u + \kappa \geq 0$ on the support of u_{τ}^{+} ; recall that $\kappa \geq 0$ by hypothesis. Also,

$$(123) \quad \begin{aligned} & \mathcal{A}(x, w_1) \nabla w_1 - \mathcal{A}(x, w_0) \nabla w_0 = \int_0^1 \partial_t \{\mathcal{A}(x, w_t) \nabla w_t\} dt \\ &= (u + \kappa) \left\{ \int_0^1 \mathcal{A}_z(x, w_t) \nabla w_t dt \right\} + \left\{ \int_0^1 \mathcal{A}(x, w_t) dt \right\} \nabla u \\ &= (u + \kappa) \tilde{a}(x) + \tilde{\mathbf{A}}(x) \nabla u, \quad \text{where} \\ & \tilde{a}(x) = \int_0^1 \mathcal{A}_z(x, w_t) \nabla w_t dt \quad \text{and} \quad \tilde{\mathbf{A}}(x) = \int_0^1 \mathcal{A}(x, w_t) dt. \end{aligned}$$

Using (123) in (121), using (122) to omit the term in (121) which involves the difference of f values, and using the facts that $u = u_{\tau}^{+} + \tau$ and $\nabla u = \nabla u_{\tau}^{+}$ on the support of u_{τ}^{+} yields

$$(124) \quad \int_{\Omega} \tilde{\mathbf{A}}(x) \nabla u_{\tau}^{+} \cdot \frac{\nabla u_{\tau}^{+}}{(u_{\tau}^{+} + \varepsilon)^2} \leq - \int_{\Omega} [(u_{\tau}^{+} + \tau) \tilde{a}(x)] \frac{\nabla u_{\tau}^{+}}{(u_{\tau}^{+} + \varepsilon)^2}.$$

Now, by Schwarz's inequality and the definition of $\tilde{a}(x)$,

$$\begin{aligned} & \left| \int_{\Omega} [(u_{\tau}^{+} + \tau) \tilde{a}(x)] \frac{\nabla u_{\tau}^{+}}{(u_{\tau}^{+} + \varepsilon)^2} \right| \\ &= \left| \int_{\Omega} \left[(u_{\tau}^{+} + \tau) \left\{ \int_0^1 \nabla u_{\tau}^{+} \cdot \mathcal{A}_z(x, w_t) \nabla w_t dt \right\} \right] \frac{1}{(u_{\tau}^{+} + \varepsilon)^2} \right| \\ &\leq \alpha \int_{\Omega} \left\{ \int_0^1 \frac{|\mathcal{A}_z(x, w_t) \nabla u_{\tau}^{+}|^2}{k^{*}(x, w_t)} dt \right\} \frac{1}{(u_{\tau}^{+} + \varepsilon)^2} \\ &\quad + \frac{C}{\alpha} \int_{\Omega} \left\{ \int_0^1 k^{*}(x, w_t) |\nabla w_t|^2 dt \right\} \frac{(u_{\tau}^{+} + \tau)^2}{(u_{\tau}^{+} + \varepsilon)^2}. \end{aligned}$$

In fact, in the last four integrations as well as those below, the domain of integration can be restricted to the compact subset $\text{supp } u_{\tau}^{+}$ of Ω . Then, since $k^{*}(x, w_t) |\nabla w_t|^2 \leq |\nabla_{\sqrt{\mathcal{A}}, w_t} w_t|^2$ due to (9), by assuming that $\tau \leq \varepsilon$ and applying (16) to the first term on the right, we obtain

$$\begin{aligned} & \left| \int_{\Omega} [(u_{\tau}^{+} + \tau) \tilde{a}(x)] \frac{\nabla u_{\tau}^{+}}{(u_{\tau}^{+} + \varepsilon)^2} \right| \\ &\leq \alpha B_{\mathcal{A}}^2 \int_{\Omega} \frac{\left\{ \int_0^1 \nabla u_{\tau}^{+} \cdot \mathcal{A}(x, w_t) \nabla u_{\tau}^{+} dt \right\}}{(u_{\tau}^{+} + \varepsilon)^2} + \frac{C}{\alpha} \int_{\Omega} \left\{ \int_0^1 |\nabla_{\sqrt{\mathcal{A}}, w_t} w_t|^2 dt \right\} \end{aligned}$$

$$\leq \alpha B_{\mathcal{A}}^2 \int_{\Omega} \frac{\nabla u_{\tau}^+ \cdot \tilde{\mathbf{A}}(x) \nabla u_{\tau}^+}{(u_{\tau}^+ + \varepsilon)^2} + \frac{C}{\alpha},$$

where we used that $w_t \in H_{\mathcal{X}}^{1,2}(\Omega)$ for $0 \leq t \leq 1$, and hence the constant C is independent of τ and ε . Taking $\alpha = 1/2B_{\mathcal{A}}^2$, combining with (124) and absorbing into the left gives

$$(125) \quad \int_{\Omega} \tilde{\mathbf{A}}(x) \nabla u_{\tau}^+ \cdot \frac{\nabla u_{\tau}^+}{(u_{\tau}^+ + \varepsilon)^2} \leq CB_{\mathcal{A}}^2, \quad 0 < \tau \leq \varepsilon.$$

From (9) and the fact that $k^1(x, z) = 1$, we have

$$\tilde{\mathbf{A}}(x) \nabla u_{\tau}^+ \cdot \nabla u_{\tau}^+ \geq (\partial_1 u_{\tau}^+)^2.$$

Then, by (125) and the identity

$$\frac{\partial_1(u_{\tau}^+)}{u_{\tau}^+ + \varepsilon} = \frac{\partial_1\left(\frac{u_{\tau}^+}{\varepsilon} + 1\right)}{\frac{u_{\tau}^+}{\varepsilon} + 1} = \partial_1 \ln\left(\frac{u_{\tau}^+}{\varepsilon} + 1\right),$$

it follows that

$$\int_{\Omega} \left| \partial_1 \ln\left(\frac{u_{\tau}^+}{\varepsilon} + 1\right) \right|^2 dx \leq CB_{\mathcal{A}}^2, \quad 0 < \tau \leq \varepsilon.$$

Applying the one-dimensional Sobolev inequality and letting $\tau \rightarrow 0$ gives

$$\frac{1}{C \operatorname{diam} \Omega} \int_{\Omega} \left| \ln\left(\frac{u^+}{\varepsilon} + 1\right) \right|^2 dx \leq \int_{\Omega} \left| \partial_1 \ln\left(\frac{u^+}{\varepsilon} + 1\right) \right|^2 dx \leq CB_{\mathcal{A}}^2$$

uniformly in $\varepsilon > 0$. Since u is continuous in $\bar{\Omega}$, it follows that $u^+ = 0$ in Ω , so that $u \leq 0$ in Ω . Hence $w_1 \leq w_0 + \kappa$ in Ω as desired.

Now, for the general case, assume that $w_0, w_1 \in (H_{\mathcal{X}}^{1,2}(\Omega) \cup \mathcal{C}^{\infty}(\Omega)) \cap \mathcal{C}^0(\bar{\Omega})$ and satisfy the hypotheses of the lemma. Consider a family $\{\Omega_{\varepsilon}\}_{\varepsilon>0}$ of open sets such that $\Omega_{\varepsilon} \nearrow \Omega$ and

$$0 < \inf \{ \operatorname{dist}(x, \partial\Omega) : x \in \partial\Omega_{\varepsilon} \} < \varepsilon.$$

Then $\Omega_{\varepsilon} \Subset \Omega$ for all $\varepsilon > 0$, and the function $\mu(\varepsilon)$ defined for $\varepsilon > 0$ by $\mu(\varepsilon) = \max_{\partial\Omega_{\varepsilon}} (w_1 - w_0 - \kappa)$ satisfies $\lim_{\varepsilon \rightarrow 0} \mu(\varepsilon) \leq 0$.

Since $w_0, w_1 \in (H_{\mathcal{X}}^{1,2}(\Omega) \cup \mathcal{C}^{\infty}(\Omega))$ and $\Omega_{\varepsilon} \Subset \Omega$, it follows that $w_0, w_1 \in H_{\mathcal{X}}^{1,2}(\Omega_{\varepsilon})$ for each $\varepsilon > 0$. Moreover, $w_0 + \kappa + \mu(\varepsilon) \geq w_1$ on $\partial\Omega_{\varepsilon}$. By the previous case,

$$w_0 + \kappa + \mu(\varepsilon) \geq w_1 \quad \text{in } \Omega_{\varepsilon}.$$

The lemma now follows by letting $\varepsilon \rightarrow 0^+$. \square

7.4. Barriers for the Dirichlet problem. In this section, we construct barrier functions for continuous weak solutions of the Dirichlet problem in a smooth, strictly convex domain. An interesting aspect of these barriers is that even though they are specialized to a particular solution w , they depend only on the modulus continuity of w .

Lemma 7.6. *Let $\Phi \Subset \Omega$ be a strongly convex domain. Let $\bar{r} = \operatorname{diam} \Phi$ and ω be a concave, strictly increasing function with $\omega \in \mathcal{C}^0([0, \bar{r}]) \cap \mathcal{C}^2((0, \bar{r}))$ and $\omega(0) = 0$. Suppose \mathcal{A} satisfies (16) and γ is of subunit type with respect to \mathcal{A} . For $m \in \mathbb{R}$, define a differential operator \mathcal{L}_m by*

$$\mathcal{L}_m h = \operatorname{div} \mathcal{A}(x, h(x) + m) \nabla h.$$

Then for every $x_0 \in \partial\Phi$ and all positive constants η, ν, m_0, K , there exists a neighborhood \mathcal{N} of x_0 and a function $h \in \mathcal{C}^0(\bar{\mathcal{N}}) \cap \mathcal{C}^{\infty}(\mathcal{N})$ such that $\mathcal{N} \subset \{|x - x_0| < \eta\}$ and

$$\begin{cases} h(x_0) = 0, \\ h(x) \leq -\omega(|x - x_0|), & x \in \Phi \cap \mathcal{N}, \\ h(x) \leq -\nu, & x \in \Phi \cap \partial\mathcal{N}, \\ \mathcal{L}_m h \geq K, & x \in \Phi \cap \mathcal{N}, \quad |m| \leq m_0, \\ \Delta h = \sum_{i=1}^n \partial_i^2 h > 0, & x \in \Phi \cap \mathcal{N}. \end{cases}$$

Proof Let $x_0 \in \partial\Phi$. By translation, we may assume that $x_0 = 0$. By using a rotation $\Theta = (\theta_{ij})_{i,j=1}^n$, we may represent $\partial\Phi$ locally as $y = \Theta x$ so that for $y' = (y_1, \dots, y_{n-1})$ and positive κ_0, r_0 depending on $\partial\Phi$, we have

$$(126) \quad \kappa_0 |y'|^2 \leq y_n, \quad (y', y_n) \in \partial\Phi, \quad |y'| < r_0.$$

By hypothesis, ω satisfies

$$(127) \quad \omega(r) \geq a_0 r \quad \text{if } r \in [0, \bar{r}], \text{ where } a_0 = \frac{\omega(\bar{r})}{\bar{r}},$$

$$(128) \quad \liminf_{r \rightarrow 0^+} \omega'(r) \geq a_0 > 0,$$

$$(129) \quad \omega''(r) \leq 0 \quad \text{if } r \in (0, \bar{r}].$$

For fixed $\alpha_0 \in (0, 1]$, set

$$(130) \quad r_0 = \begin{cases} \omega^{-1}(\alpha_0) & \text{if } \omega(\bar{r}) > \alpha_0 \\ \bar{r} & \text{otherwise,} \end{cases}$$

i.e., r_0 is the largest $r \in (0, \bar{r}]$ with $\omega(r) \leq \alpha_0$. Letting $\psi(r) = \sqrt{\omega(r)}$ for $0 < r \leq r_0$, we have by (127)–(129) and since $\lim_{r \rightarrow 0^+} \omega(r) = 0$ that

$$(131) \quad 1 \geq \sqrt{\alpha_0} \geq \psi(r) \geq \omega(r),$$

$$(132) \quad -\psi'' = -\frac{2\omega''\omega - (\omega')^2}{4\omega^{3/2}} \geq \frac{(\omega')^2}{4\omega^{3/2}} = \frac{(\psi')^2}{\psi} > 0,$$

$$(133) \quad \psi(r) \geq a_1 \sqrt{r}, \quad \text{where } a_1 = \sqrt{a_0},$$

$$(134) \quad \lim_{r \rightarrow 0^+} \psi'(r) = \lim_{r \rightarrow 0^+} \frac{\omega'(r)}{2\sqrt{\omega(r)}} = +\infty.$$

For $t > 0$, let $\mathcal{N}_t = \{y_n < t\} \cap \{|y| < r_0\}$. Because of (9) (recall that $k^1 = 1$) and continuity of \mathcal{A} , there exist $1 \leq \ell \leq n$ and $c_1 > 0$ such that for all small t ,

$$(135) \quad \vec{\theta}_\ell \mathcal{A}(y, z) (\vec{\theta}_\ell)' \geq c_1 > 0, \quad y \in \partial\Phi \cap \mathcal{N}_t, \quad |z| \leq 2m_0.$$

Here $\vec{\theta}_\ell$ is the ℓ^{th} column of Θ . For $m_1 > 0$ and $0 < t_1 \leq 1$ to be determined, define

$$(136) \quad h(y) = -2\psi(\sqrt{\rho y_n}) + m_1 \frac{y_\ell^2}{2} + \frac{1}{\ln y_n}, \quad y \in \mathcal{N}_{t_1},$$

where $\rho = \left(\kappa_0^{-\frac{1}{2}} + 1\right)^2$. For t_1 small enough h is well-defined in \mathcal{N}_{t_1} and extends continuously to $\overline{\mathcal{N}_{t_1}}$ with

$$(137) \quad h \in C^\infty(\mathcal{N}_{t_1}) \cap C^0(\overline{\mathcal{N}_{t_1}}), \quad h(0) = 0.$$

From (126),

$$(138) \quad \begin{aligned} \sqrt{\rho y_n} &= \left(\kappa_0^{-\frac{1}{2}} + 1\right) \sqrt{y_n} = \sqrt{\frac{y_n}{\kappa_0}} + \sqrt{y_n} \\ &\geq |y'| + y_n \geq |y|, \end{aligned}$$

where we used that $y_n \leq 1$ in \mathcal{N}_{t_1} .

Also, by taking t_1 small enough depending on κ_0, a_1, m_1 , we obtain from (126) and (133) that

$$\begin{aligned} m_1 \frac{y_\ell^2}{2} - \psi(\sqrt{\rho y_n}) &\leq m_1 \frac{y_n}{2\kappa_0} - a_1 \sqrt[4]{\rho y_n} \\ &\leq m_1 \frac{\sqrt[4]{y_n}}{2\kappa_0} \left(t_1^{3/4} - \frac{2\kappa_0 a_1 \sqrt{\kappa_0^{-\frac{1}{2}} + 1}}{m_1} \right) \leq 0. \end{aligned}$$

Then, using (138), the fact that ψ is increasing, the definition of ψ and (131), we get

$$\begin{aligned}
 h(y) + \omega(|y|) &= -2\psi(\sqrt{\rho y_n}) + m_1 \frac{y_\ell^2}{2} + \frac{1}{\ln y_n} + \omega(|y|) \\
 &\leq \omega(|y|) - \psi(\sqrt{\rho y_n}) \\
 &\leq \psi^2(|y|) - \psi(|y|) < 0.
 \end{aligned}
 \tag{139}$$

Now set

$$G_n(y) = \left(\frac{\rho}{y_n}\right)^{\frac{1}{2}} \psi'(\sqrt{\rho y_n}) + \frac{1}{(\ln y_n)^2 y_n} > 0,$$

and write $h_j = \frac{\partial h}{\partial y_j}$, $h_{ij} = \frac{\partial^2 h}{\partial y_i \partial y_j}$. If δ_{ij} denotes the Kronecker delta, we have

$$h_n(y) = -G_n + \delta_{\ell n} m_1 y_n, \quad h_\ell(y) = m_1 y_\ell - \delta_{\ell n} G_n \tag{140}$$

$$h_i(y) = 0 \quad \text{for } i \neq n, \ell \tag{141}$$

$$h_{\ell\ell}(y) = m_1 + \delta_{\ell n} \frac{1}{2y_n} [G_n(y) - \rho\psi''(\sqrt{\rho y_n})] \tag{142}$$

$$+ \delta_{\ell n} \frac{1}{(\ln y_n)^2 y_n^2} \left(\frac{2}{\ln y_n} + \frac{1}{2} \right) \tag{143}$$

$$h_{nn}(y) = \frac{1}{2y_n} [G_n(y) - \rho\psi''(\sqrt{\rho y_n})] \tag{144}$$

$$+ \frac{1}{(\ln y_n)^2 y_n^2} \left(\frac{2}{\ln y_n} + \frac{1}{2} \right) + \delta_{\ell n} m_1$$

$$h_{ii}(y) = 0 \quad \text{for } i \neq n, \ell, \quad h_{ij}(y) = 0 \quad \text{for } i \neq j. \tag{145}$$

In particular, for t_1 small enough,

$$\Delta h(y) = \frac{1}{2y_n} [G_n(y) - \rho\psi''(\sqrt{\rho y_n})] + \frac{1}{(\ln y_n)^2 y_n^2} \left(\frac{2}{\ln y_n} + \frac{1}{2} \right) + m_1 > 0 \tag{146}$$

in \mathcal{N}_{t_1} because of (132). Moreover, by (132) and the formulas for derivatives of h ,

$$\begin{aligned}
 |h_\ell(y)|^2 &\leq 2m_1^2 y_\ell^2 + 2\delta_{\ell n} G_n^2 \\
 &\leq c \max \left\{ m_1 y_\ell^2, \delta_{\ell n} \psi(\sqrt{\rho y_n}), \frac{\delta_{\ell n}}{(\ln y_n)^2} \right\} h_{\ell\ell}(y),
 \end{aligned}
 \tag{147}$$

$$\begin{aligned}
 |h_n(y)|^2 &\leq 2G_n^2 + 2\delta_{\ell n} m_1^2 y_n^2 \\
 &\leq c \max \left\{ \psi(\sqrt{\rho y_n}), \frac{1}{(\ln y_n)^2}, \delta_{\ell n} m_1 y_n^2 \right\} h_{nn}(y).
 \end{aligned}
 \tag{148}$$

Let $H(x) = h(\Theta x)$ and $\tilde{H}(x) = H(x) + m = h(\Theta x) + m$ for fixed $m \in [-m_0, m_0]$. Using Θ^t to denote the transpose of Θ , we then have $\nabla H(x) = \Theta^t(\nabla h)(\Theta x)$, $H_j(x) = \sum_k \theta_{jk} h_k(\Theta x)$ and

$$\begin{aligned}
 H_{ij}(x) &= \sum_k \theta_{jk} \sum_\lambda \theta_{i\lambda} h_{k\lambda}(\Theta x) = \sum_k \theta_{jk} \theta_{ik} h_{kk}(\Theta x) \\
 &= \theta_{j\ell} \theta_{i\ell} h_{\ell\ell}(\Theta x) + \theta_{jn} \theta_{in} h_{nn}(\Theta x),
 \end{aligned}$$

where only one of the last two terms appears in case $\ell = n$.

Setting $\mathcal{A}(x, \tilde{H}(x)) = (a_{ij}(x))$ and letting $[\operatorname{div} \mathcal{A}(x, \tilde{H}(x))] = [\operatorname{div}(a_{ij}(x))]$ denote the vector whose components are the divergence of the columns of $(a_{ij}(x))$, we obtain

$$\begin{aligned}
 \mathcal{L}_m h &= \operatorname{div} \left[\mathcal{A} \left(x, \tilde{H}(x) \right) \nabla H(x) \right] = \operatorname{div} \left(\sum_i a_{ij}(x) H_i(x) \right)_{j=1, \dots, n} \\
 &= [\operatorname{div}(a_{ij}(x))] \cdot \nabla H(x) + \sum_{i,j} a_{ij}(x) H_{ij}(x)
 \end{aligned}$$

$$\begin{aligned}
&= [\operatorname{div}(a_{ij}(x))] \cdot \Theta^t(\nabla h)(\Theta x) + h_{\ell\ell}(\Theta x) \sum_{i,j} a_{ij}(x) \theta_{j\ell} \theta_{i\ell} + h_{nn}(\Theta x) \sum_{i,j} a_{ij}(x) \theta_{jn} \theta_{in} \\
&= [\operatorname{div}(a_{ij}(x))] \cdot \Theta^t(\nabla h)(\Theta x) + h_{\ell\ell}(\Theta x) \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t + h_{nn}(\Theta x) \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t.
\end{aligned}$$

Here $\vec{\theta}_j^t$ denotes the j^{th} column of Θ and $\vec{\theta}_j$ denotes the j^{th} column of Θ rewritten as a row vector, and in the last two equalities, only one of the second and third terms on the right appears in case $\ell = n$.

Now recall that $h_i = 0$ if $i \neq \ell, n$. Then the first term on the right of the last equality equals

$$\begin{aligned}
&h_\ell(\Theta x) \sum_k \left(\sum_i \frac{\partial}{\partial x_i} a_{ik}(x) \right) \theta_{k\ell} + h_n(\Theta x) \sum_k \left(\sum_i \frac{\partial}{\partial x_i} a_{ik}(x) \right) \theta_{kn} \\
&= h_\ell(\Theta x) \left[\operatorname{div} \mathcal{A}(x, \tilde{H}(x)) \right] \cdot \vec{\theta}_\ell^t + h_n(\Theta x) \left[\operatorname{div} \mathcal{A}(x, \tilde{H}(x)) \right] \cdot \vec{\theta}_n^t,
\end{aligned}$$

where as usual only one of the two terms on the right appears in case $\ell = n$.

Altogether,

$$\begin{aligned}
\mathcal{L}_m H(x) &= h_\ell(\Theta x) \left[\operatorname{div} \mathcal{A}(x, \tilde{H}(x)) \right] \cdot \vec{\theta}_\ell^t + h_n(\Theta x) \left[\operatorname{div} \mathcal{A}(x, \tilde{H}(x)) \right] \cdot \vec{\theta}_n^t \\
&\quad + h_{\ell\ell}(\Theta x) \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t + h_{nn}(\Theta x) \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t,
\end{aligned}$$

where on the right side, only one of the first two terms and one of the second two terms appears in case $\ell = n$.

By direct computation, we have

$$\begin{aligned}
\left[\operatorname{div} \mathcal{A}(x, \tilde{H}(x)) \right] \cdot \vec{\theta}_r^t &= \sum_{i=1}^n \mathcal{A}_i^i(x, \tilde{H}(x)) \cdot \vec{\theta}_r^t + h_\ell(\Theta x) \vec{\theta}_\ell \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_r^t \\
&\quad + h_n(\Theta x) \vec{\theta}_n \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_r^t,
\end{aligned}$$

where \mathcal{A}^i denotes the i^{th} -column of \mathcal{A} and $\mathcal{A}_i^i = \partial_i \mathcal{A}^i$, and where only one of the last two terms on the right side appears in case $\ell = n$. Substituting this in the formula above for $\mathcal{L}_m H$ gives

$$\begin{aligned}
\mathcal{L}_m H(x) &= \mathcal{L}_m[h(\Theta x)] = h_\ell(\Theta x) \sum_{i=1}^n \mathcal{A}_i^i(x, \tilde{H}(x)) \cdot \vec{\theta}_\ell^t + h_\ell(\Theta x)^2 \vec{\theta}_\ell \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_\ell^t \\
&\quad + h_\ell(\Theta x) h_n(\Theta x) \vec{\theta}_n \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_\ell^t + h_\ell(\Theta x) h_n(\Theta x) \vec{\theta}_\ell \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_n^t \\
&\quad + h_n(\Theta x) \sum_{i=1}^n \mathcal{A}_i^i(x, \tilde{H}(x)) \cdot \vec{\theta}_n^t + h_n(\Theta x)^2 \vec{\theta}_n \mathcal{A}_z(x, \tilde{H}(x)) \vec{\theta}_n^t \\
&\quad + h_{\ell\ell}(\Theta x) \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t + h_{nn}(\Theta x) \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t,
\end{aligned}$$

and in case $\ell = n$, only the first, second and seventh terms on the right appear. By (12), the sum of the first and fifth terms on the right is at most

$$cB_{\mathcal{A}}^2 + \frac{h_\ell(\Theta x)^2}{4} \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t + \frac{h_n(\Theta x)^2}{4} \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t,$$

and the sum of the second, third, fourth and sixth terms is bounded by

$$cB_{\mathcal{A}} \left(h_\ell(\Theta x)^2 \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t + h_\ell(\Theta x)^2 \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t \right),$$

where $B_{\mathcal{A}} = B_{\mathcal{A}}(\Phi, m_0)$. Using these estimates, we obtain

$$\begin{aligned}
\mathcal{L}_m H(x) &\geq -cB_{\mathcal{A}}^2 + \left[h_{\ell\ell}(\Theta x) - \left(cB_{\mathcal{A}} + \frac{1}{4} \right) h_\ell(\Theta x)^2 \right] \vec{\theta}_\ell \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_\ell^t \\
&\quad + \left[h_{nn}(\Theta x) - \left(cB_{\mathcal{A}} + \frac{1}{4} \right) h_n(\Theta x)^2 \right] \vec{\theta}_n \mathcal{A}(x, \tilde{H}(x)) \vec{\theta}_n^t.
\end{aligned}$$

From (147) and (148), by taking t_1 small enough (depending on κ_0 , m_1 , and $B_{\mathcal{A}}$), we obtain

$$\mathcal{L}_m H(x) \geq -cB_{\mathcal{A}}^2 + \frac{h_{\ell\ell}(\Theta x)}{2} \bar{\theta}_{\ell} \mathcal{A}\left(x, \tilde{H}(x)\right) \bar{\theta}_{\ell}^t.$$

Then from (135) and the fact that $h_{\ell\ell} \geq m_1$,

$$(149) \quad \mathcal{L}_m h \geq -cB_{\mathcal{A}}^2 + \frac{c_1 m_1}{2} > K$$

by taking m_1 large enough.

Finally, note that $y_{\ell}^2 \leq y_n/\kappa_0$ by (126). Then if $x \in \Phi \cap \partial\mathcal{N}_{t_1}$,

$$(150) \quad \begin{aligned} h(y) &= -2\psi(\sqrt{\rho y_n}) + m_1 \frac{y_{\ell}^2}{2} + \frac{1}{\ln y_n} \\ &\leq \frac{m_1}{2\kappa_0} t_1^2 + \frac{1}{\ln t_1} \leq -\nu \end{aligned}$$

by taking t_1 small enough. The condition $\mathcal{N} \subset \{|x - x_0| < \eta\}$ can then be met by taking t_1 even smaller. Lemma 7.6 now follows from (137), (139), (146), (149) and (150).

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